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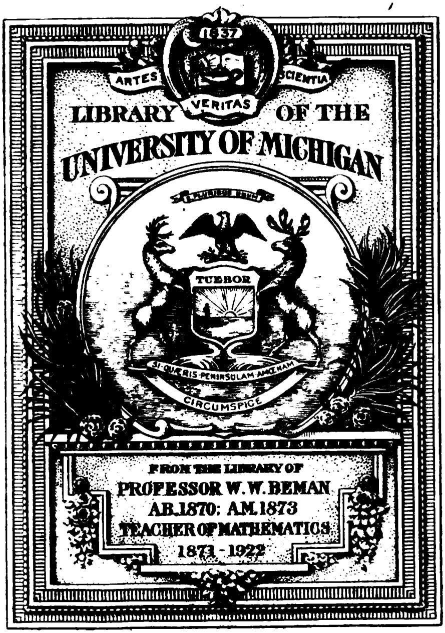

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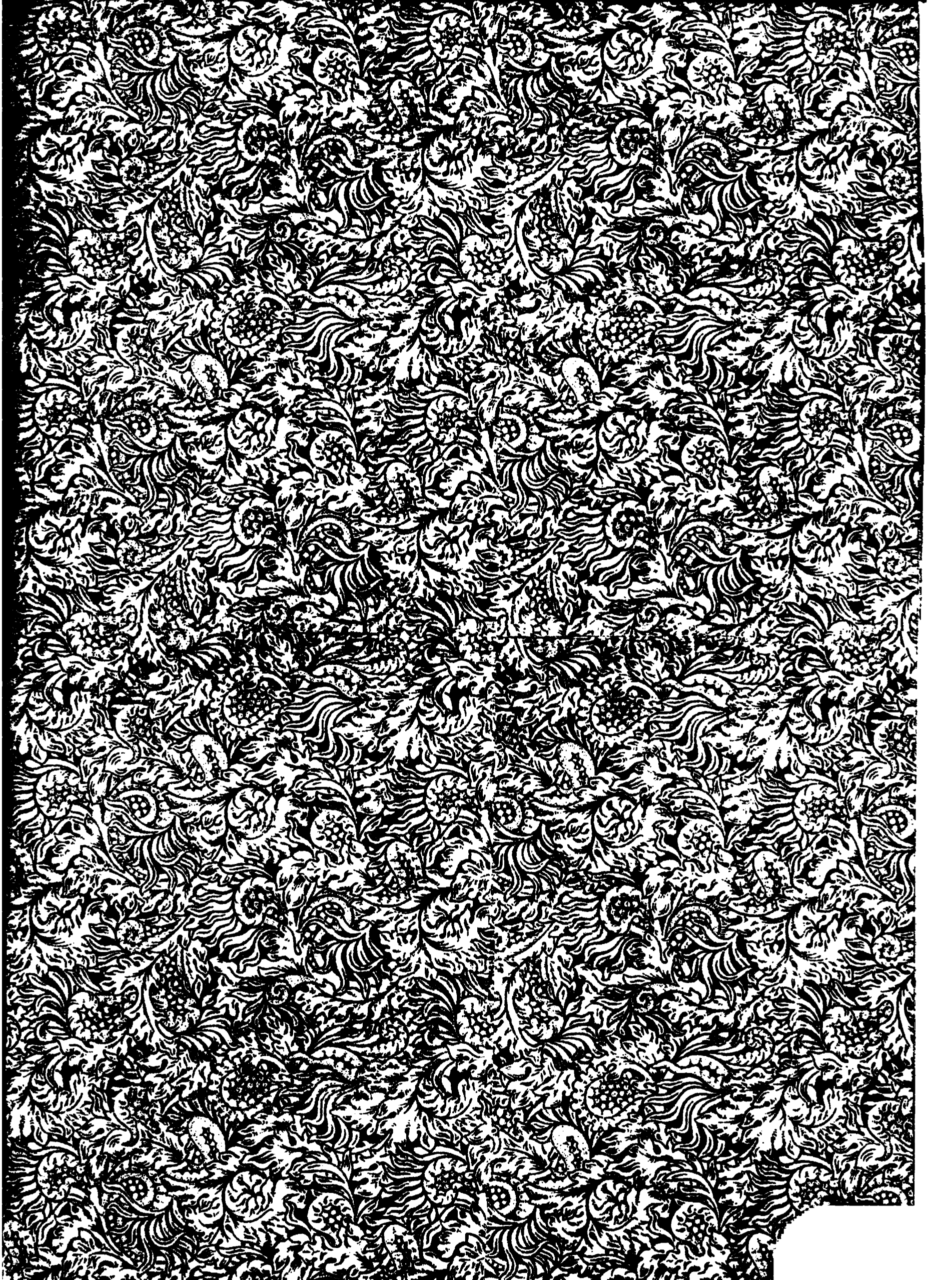

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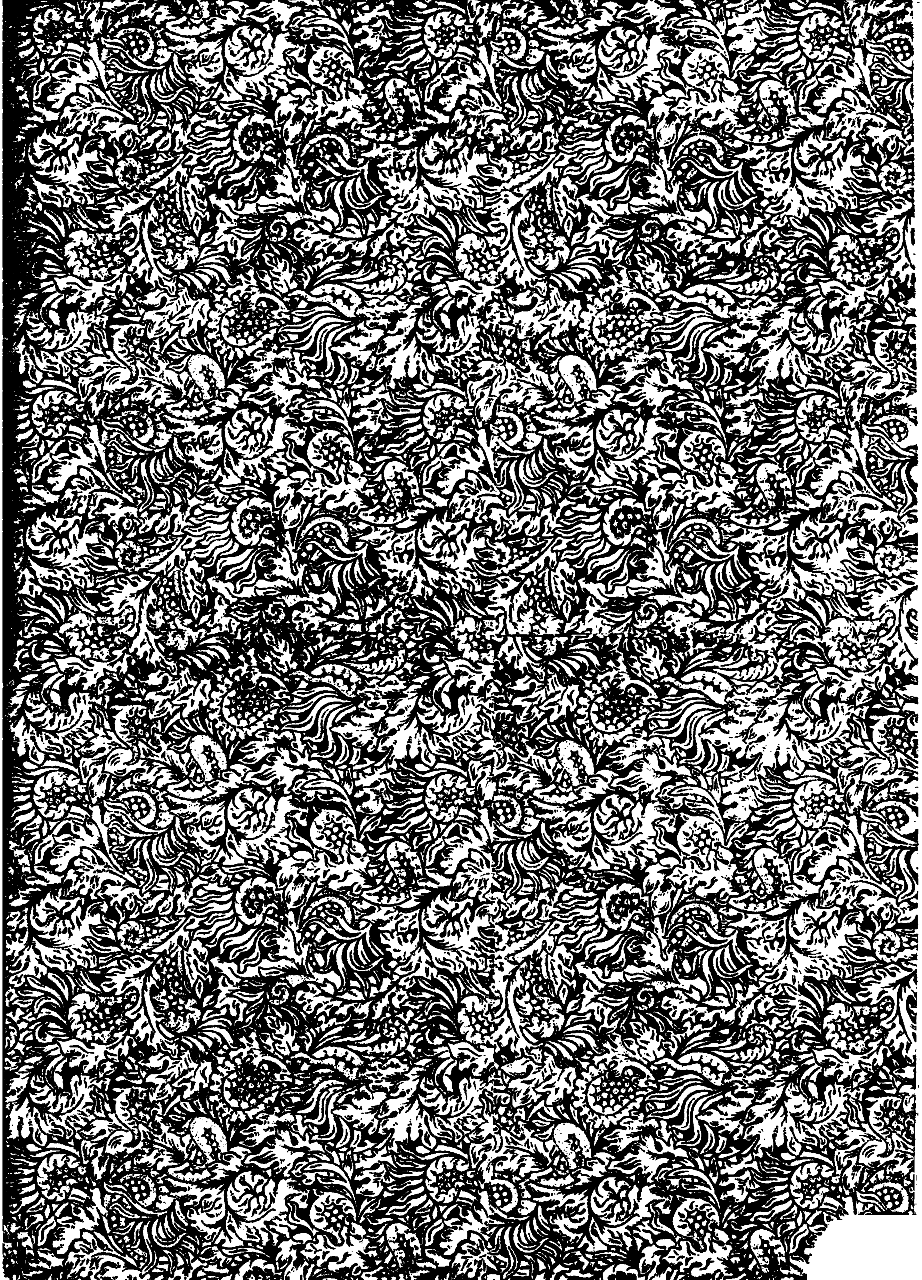

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GEOMETRY

IN

THE SECONDARY SCHOOL.

1889 —

BY

J. W. MacDONALD,

Principal of the Stoneham (Mass.) High School.

"Sçavoir par cœur n'est pas sçavoir." — *Montaigne.*

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PREFACE.

IF it seem to any one presumptuous that one who lays no claim to being a profound mathematician should venture to publish a geometry and a treatise on how it ought to be taught in secondary schools, let me say, in explanation, that a profoundly learned and exhaustive treatise on a subject, and a practical school-book designed with a view to the development of intellectual power as well as for furnishing information, are very different things. For producing the former, profound special learning is essential; for the latter, less special learning may suffice, provided it is supplemented by a practical knowledge of psychology. Even an average scholar knows a great deal more than ought to be put into an elementary school-book, and the profound specialist is strongly inclined to yield to the common passion to tell, after once getting started, all he knows. This has worked great injury to many kinds of text-books, especially those in mathematics and English literature; for it has produced text-books that usurp the function of the teacher, and, except memorizing, of the pupil also. A reform in this particular is greatly needed, and such a reform this treatise is designed to further.

In closing, I wish to acknowledge my obligations to my friend, Mr. H. N. Wheeler, to whose hints, encouragement, and assistance I feel greatly indebted.

J. W. M.

STONEHAM, July 28, 1889.

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GEOMETRY IN THE SECONDARY SCHOOL.

CHAPTER I.

THE PURPOSE OF GEOMETRY IN THE SECONDARY SCHOOL. — HOW TO ATTAIN IT.

THE little book* for which this treatise is partly intended as a plea, is the outcome of careful reflection on the purpose of studying geometry, and of observation of the common methods. I am convinced that the purpose and the methods do not connect. While, by general assent, the chief purpose of the study is to train the reasoning faculties, the methods, measured by psychological laws, are not adapted to effect that result. The way and the only way to acquire facility in doing anything mentally or physically, is to do it, and doing something else, whatever the semblance, will not train the faculty. A boy will never learn to skate by memorizing directions; no more will he become a reasoner by memorizing the reasoning processes of others.

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* Principles of Plane Geometry. By J. W. MacDonald. Published by Allyn & Bacon, Boston, 1889.

The issue may be briefly put: Does a student in geometry need the assistance the prevailing text-books in that science furnish him? Does he need to have the propositions all illustrated by figures, and the demonstrations worked out for him? Is he incapable of illustrating a clearly stated proposition, and thinking out a proof founded on principles he has already learned? For myself, *usu magistro*, I emphatically answer *no.* With a few suggestions, under the guidance of a skilful teacher, the average student will think out for himself nine tenths of what the text-book tells him. But the temptation to memorize must be taken from him, and the propositions of geometry placed before him in a well-arranged syllabus. The prevailing text-book in geometry, with its superfluous and mischievous explanations and demonstrations, is the very centre and stronghold of the "cram" method. Besides, it discountenances self-reliance, begets a dwarfing reverence for authority, and tends to make the student a mere imitator.

I do not underestimate the value of example and imitation. Example and imitation are valuable in stimulating independent individual effort, when intermixed in the proportion of from one to ten per cent. But when they constitute from ninety to a hundred per cent of the entire effort of the pupil, as is commonly the case in studying geometry, it is expanding their function altogether too much. The defect in the prevailing method of studying geometry may be most clearly seen by comparing it with the method in another branch of which the purpose is in most respects different; history, for example. So far from these studies being supplementary in their intellectual effect, it will, I am confident, be found that the mental processes called into activity are essentially the same in both.

First, let us observe the work of the pupil in studying history. The lesson, we will suppose, is from early American history, say the intercolonial war known as King William's War, and the text on which the pupil is to prepare himself reads somewhat thus:—

"*Cause.* War having broken out in Europe between England and France, their colonies in America took up the quarrel. The Indians of

Canada and Maine aided the French, and the Iroquois assisted the English.

"*Attacks upon the Colonists.* War parties of the French and Indians coming down on their snow-shoes from Canada, through the forest in the depth of winter, fell upon the exposed settlements of New York and New England. The most horrible barbarities were committed. Exposed settlements were attacked, and men, women, and children were either tomahawked or carried into captivity."

Just here there is, perhaps, inserted an illustrative wood-cut, representing a huge savage in feathers and paint, with one hand clutching by the hair a kneeling woman who is clasping a child to her bosom, while with the other hand he swings aloft a tomahawk.

> "Implicuitque comam læva, dextraque coruscum
> Extulit — ensem."

After the cut the text goes on to give an account of the capture of Schenectady and the massacre of its inhabitants, and so on.

Now for the purpose of our investigation, let us take a studious pupil who faithfully prepares himself on the lesson assigned, and follow him in the work of preparation. What are his mental processes?

First, he is exercising his powers of attention. This, however, is a general requirement in all volitional operations, bad as well as good, and will be found equally exerted in the preparation of the geometry lesson. It may therefore be eliminated from our investigation, which chiefly seeks to discover the peculiar mental processes evoked by each study. The same may be said of the complex mental processes by which the printed symbols of the page are translated into thoughts. Passing over these also, we shall find that the chief process going on in the pupil's mind is memorizing. I do not mean an unintelligent memorizing of the words, but memorizing the facts, memorizing in the best sense of the word that the preparation of a history lesson would justify. This, too, is in some respects a common process, but the reason why it may be dwelt upon here, is because it seems to be the peculiarly active process exercised in the elementary study of history. The pupil is not obliged to compare, classify, and deduce, and probably does not. He only aims so to fix in his memory the facts narrated that he may be able to restate them in the recitation, or, at most, to retain them as additions to his knowledge.

The wood-cut to which I have alluded could not fail to attract his notice, and he would undoubtedly observe that it was in harmony with the statements of the text. This would involve a rudimentary exercise of the faculties of observation and comparison. Maps would have a similar effect. This practically completes the list of mental processes involved in the preparation of the lesson: namely, memorizing for the most part, and to a slight degree observing and comparing. When the boy comes into recitation, a skilful teacher may make use of what he has learned to call out other mental activities, by such questions as: Why were the events narrated in the lesson natural results? How as to their moral aspect would they be viewed by the hostile peoples? How by the impartial historian? Can you assign to each party its due proportion of just and unjust conduct? What were the immediate consequences? How did these events contribute (answer, of course, to be held in suspense till the pupil can deduce it from his own reading) to subsequent political and social developments? But though a skilful teacher can call out, as I have said, other mental activities by such questions as these (that is, if some historical nuisance has not taken it into his head to write a school history explaining all these points), yet all teachers are not skilful, and so I present them only as possibilities connected with the study of history.

And now let us follow our pupil through a lesson in geometry. He opens his book and commits to memory the following proposition: —

" If two straight lines intersect each other, the vertical angles are equal."

Having memorized the theorem, he proceeds with the demonstration. Now one would suppose that the boy ought to know what "straight lines" are, and what " intersect" means, and what " vertical angles" are, but not so does the book before me assume, for it goes on to elucidate these points as follows: —

" Let the lines A B and C D intersect at the point E; then the angles A E C and D E B are equal." Here there is inserted a cut representing the two lines crossing each other and lettered as given, so that the pupil may *see* the author's explanation. Glancing at this as he proceeds, our student goes on to prove the theorem by the following line of reasoning, all written out for him by the kindness of the author:—

"For the angles A E C + A E D are equal to two right angles"—(observe that the pupil has just "demonstrated" this, and yet when he comes to it again, he is not supposed to know it,—a supposition, by the way, considering the way in which he did "demonstrate" it, that is probably correct. At any rate there is inserted here a reference to the previous proof, if not on the supposition that the pupil in learning it has not acquired sufficient command of it to use it, then why?)—"The angles A E D + D E B," continues the text, "are for the same reason equal to two right angles, therefore the angles A E C + A E D = A E D + D E B. Axiom 1." (Observe again the assumption of ignorance.) "Subtracting from both members of the equation the common angle A E D, it leaves the angle A E C equal to the angle D E B. Axiom 3." It does seem as if the author might have left this last little point for the pupil to have applied for himself, but perhaps his judgment as to what a pupil trained by his method is capable of should not be questioned. The pupil goes over this explanation several times with a view, if he is wise, to the recitation only; for as a fragment of knowledge it is next to worthless. So far the mental faculties exerted have been memorizing for the most part and, in a rudimentary way, observing and comparing.

In recitation his chief effort will be to recall and repeat the argument he has memorized, and no amount of skill on the part of the teacher can call out much more than a glib recitation.

For any one to say that the student who has gone through geometry in this way has succeeded in the study, while the student who has taken it in the way I advocate, has failed, even if he has only gone over a half, a quarter, or a tenth the ground, is to betray utter ignorance of the object of the study, and the laws of psychology. To nine out of every ten students, almost the entire value of the study is the mental training it imparts. A fellow-student of mine in college used to claim that he got more mental discipline scheming to cut recitations, than the rest of us did striving to perfect our recitations in the way we were taught. I used to argue the question with him, but I did not then know what education was. If the student has not acquired the power to fix his mind on a subject, to think into it and around it, to catch its secrets, to comprehend its truths, and to draw inferences from them, he has gained nothing at all by the study. On this point I am glad to be able to cite so eminent an authority in corroboration as Mr. J. G. Fitch, who says: —

> "Our attention to-day has been necessarily confined to the consideration of a rational way of treating arithmetic, the one department of mathematics with which, in a school, the teacher is first confronted. But the same general design should be in the mind of the teacher through geometry, algebra, trigonometry, the calculus, and all the later stages of mathematical teaching. While constantly testing the success of his pupils by requiring problems to be worked, he will, nevertheless, feel that the solution of problems is not the main object of this part of his school discipline, but rather the insight into the meaning of processes, and the training in logic. If algebra and geometry do not make the student a clearer and more accurate and more consecutive thinker, they are worth nothing. And in the new revolt

against the long supremacy of Euclid, as represented in the Syllabus of the 'Association for the Improvement of Geometrical Teaching,' the one danger we have to fear is that the demonstrative exercises will be cut up into portions too small to give the needful training in continuity of thought. Euclid, with all his faults, obliges the learner to keep his mind fixed not only on the separate truths, but also on the links by which a long succession of truths are held together. It is well to simplify the science of geometry, and to arrange its various theorems in a truer order. But since it is not geometry, but the mental exercise required in understanding geometry, which the student most wants to acquire, a system of teaching which challenged less of fixed attention, and substituted shorter processes for long, would possibly prove rather a loss than a gain."—*Lectures on Teaching, American Edition, page* 310.

The question, then, is simply this: in which way will pupils cultivate these powers best; by tracing out and memorizing the processes of others, or by thinking out these processes for themselves?

Lest the reader may think that I have forgotten about the "original demonstrations," on which so much reliance is placed in connection with geometry, I here assure him that I have given them a very careful consideration; and while I approve of this work to a considerable extent, I believe that by the prevailing methods of teaching geometry, its value is largely made nugatory. In the first place, the original demonstrations are generally plain deductions easily seen by the pupil, when the remembrance of a demonstration that he has just learned is fresh in his memory, and their solution indicates an evanescent intuition that will fade with the fading remembrance, rather than that useful intuition that springs from a trained mind. When they are in any way difficult, they are generally so accompanied with hints and suggestions by the author or the teacher, that the relative proportion of the pupil's "original" work is about that of the "half-penny worth of bread" in Falstaff's bill. The condition of things pertaining to original work in geometry, in one of the higher institutions of learning in Massachusetts, where the study is made much of, is fairly well shown by the following conversation with a student, one of the best in his class: —

"What part of your work in geometry is assigned for original demonstrations?"

"I should say about twenty-five per cent."

"How do the students generally do them?"

"Well, we find a considerable part of them in other geometries that are in the library, or brought by some one, and sometimes we get points from the older students."

"And how about the rest?"

"Oh, well, the rest are 'most always so easy, we can see right through them."

In the second place, the propositions given in geometries for original demonstration do not form a connected chain of reasoning, as do the progressive propositions of a well-arranged system of geometry. It is true, they are deduced from previously established principles, but they lead nowhere, and though interesting in themselves, it is the interest that attaches to the queer problems in a family paper, or the puzzles in an almanac. They are, therefore, not only unsuited for training the pupil in consecutive reasoning, but when introduced, as they are, into a system of geometry, they tend to distract his mind from the essential continuity, and, so far as they do this, they do more harm than good. The more enthusiastic a pupil becomes over the examples for original demonstration, when studying geometry with the usual text-book, the more he is liable to this error. I do not object to outside propositions, but the teacher should take the utmost care that the pupil understands that they are introduced "just for fun," and that they are in no sense a part of the system of mathematical principles that spring from axioms, and lead to the area of the circle, and solid geometry.

Lastly, they are introduced at the wrong period in the scholar's work. In all endeavors, it is the first step "that craves wary walking." Generally it is the earliest training and first impulses that have the most to do with giving trend to individual characteristics; and this is true not only as to moral, social, and business traits, but as to mental tendencies as well. Furthermore, as a method once adopted tends by every exercise to become a habit, so it makes a change to another method all the more difficult. Hence the importance of the initial method, especially in matters of education.

"Things ill begun make strong themselves by ill."

The initial method will largely determine the subsequent mental attitude of the pupil towards a study, no matter what may afterwards be done, as is well shown by the testimony of the student cited above. In geometry, then, the method by original demonstration should precede that of memorizing, *if it is the better*. If, I say, it is the better, and that it is better I am glad to believe there is almost a universal consensus of opinion. As an evidence of this we have the statements of eminent teachers, the requirements of the leading colleges, and the fact that all the most popular geometries have inserted copious examples in whole or in part to be worked out by the pupil. These, as I have already said, have nothing to do with a working system of geometry, and their only excuse for being is the superiority of the method by which they are supposed to be proven. It only remains, therefore, to convince teachers that this method can be as successfully applied to the propositions of the system, as to isolated ones; or, in other words, that it is as easy to teach a boy to reason for himself from the start, as after he has been spoiled by wrong habits.

The difficulty of thus convincing teachers springs from various causes.

First. Custom and the standard school geometries that lend their unanimous influence to the "cram" method. One cannot, if he stops to think, but feel surprised that a study having for its purpose the training of the reasoning powers, as all agree, should be taught so long almost unquestioned by a method that is itself contrary to reason, inasmuch as it contradicts one of the plainest principles of psychology: that memorizing or tracing out a line of reason deduced by another does not in itself develop reasoning powers.

Secondly. College requirements, by making the quantity more prominent in the teacher's mind, than the quality, lead him to take what seems at the outset the quickest way to get there.

Thirdly. A lack of confidence in the ability of the pupils, and an impatience that will not allow them time to digest the knowledge already acquired, and discover its bearings, — an impatience that impels the teacher to be always meddling. To such teachers I recommend faith.

Fourthly. A lack of qualification for the work, and especially a lack of skill in holding a pupil to a line of reasoning, without telling him what to think.

Lastly. A feeling on the part of many teachers that the pupil needs a written model. In a geometry before me as I write, I have counted the "models" put before the pupil, and I find in the five books of plane geometry about one hundred and seventy-five propositions fully explained, to say nothing of all the carefully worded definitions. I would respectfully ask if this is not carrying the "theory of models" rather too far. Besides is it too much to expect of a pupil, after all his previous study and drill in language, that in so plain a thing as the demonstration of a proposition, after the steps to it have been developed, he should construct his own model? And this he will easily do with a very little guidance by the teacher. There is always, however, a peculiar satisfaction in fighting fire with fire; and, therefore, I take considerable pleasure in quoting in this connection the advice of Prof. G. A. Wentworth, himself the author of a geometry of the kind that the advocates of models admire, and, let me add, one of the best of its kind. Prof. Wentworth in a note "To the Teacher," appended to the Preface of the first edition of his geometry and retained in the revised edition, gives this advice: —

"When the pupil is reading each book for the first time, it will be well to let him write out his proofs on the black-board in his own language; care being taken that his language be the simplest possible, that the arrangement of work be vertical (without side work), and that the figures be accurately constructed.

"This method will furnish a valuable exercise as a language lesson, will cultivate the habit of neat and orderly arrangement of work, and will allow a brief interval for deliberating on each step.

"After a book has been read in this way, the pupil should review the book and should be required to draw the figures freehand. He should state and prove the propositions orally, using a pointer to indicate on the figure every line and angle named. He should be encouraged, in reviewing each book, to do the original exercises; to state the converse of propositions; to determine from the statement, if possible, whether the converse be true or false, and if the converse be true to demonstrate it, and also to give well-considered answers to questions which may be asked him on many propositions."

It would seem, then, that on the authority of an able mathematician, the models are not to be used first, or so far as I can see, at all, for it is with me a query what is the use of all Prof. Wentworth's clear and elaborate explanations of all the chief propositions of his book, if the teacher is to follow out the advice given in the above note; but I nevertheless think that Prof. Wentworth is right *in his Preface*, and if so, then what a contemner of models! It is as if Demetrius himself had become a denouncer of idols.

An admission generally made by the adherents of the "models" is, that "to prevent the pupil from getting the demonstration by rote, they have him change the lettering." But if it is necessary to have a model to teach the boy to say, "Let the line A B intersect the line C D at the point E," is not the lettering an essential part of the model, and would not any change in them be taking undue liberties with the sacredness of the model?

To these and such other objections as teachers raise to the new method, it is difficult to make any convincing answer, because they for the most part spring from the prejudice of habit, and from the imagination. As a means of exciting a healthful scepticism regarding obstacles, I would recommend to teachers to read the objections raised in the English Parliament against granting the charter for the first railway in England; and then I would have them give the new method a fair trial, or rather I should say "the old method," for it is really a return to the earlier way of teaching geometry. At first it would not be strange if there were failures. But these failures, as a candid investigation will show, will full as often be the fault of the teacher as of the pupils. The teacher must learn largely by experience how to do, and one thing he must learn above all, and that is to avoid unnecessary chattering and meddling, — a fault in teaching without a fellow. If he will conduct two classes through plane geometry by the method I advocate, I am confident he would never again give his consent to the use of a text-book in geometry having the figures all drawn and illustrated and the demonstrations all worked out.

Let it be observed that the evil of using such books does not stop with geometry. Take the common text-book with its copious annotations for teaching English literature, and what is it but the natural consequence of the common text-book in geometry, algebra, and arithmetic? Having pursued their early studies by methods that give them no individual power, pupils must have in their subsequent studies such methods as require no individual effort.

There is, however, one real difficulty which the teacher will encounter, — a result, by the way, of the false notions of both parents and pupils as to what education really is. It will be to get the pupils as a whole to put into their daily preparation the requisite amount of study. Some of them will do this on the mere suggestion of the teacher as to what they should do and the minimum amount of time they should give to it. But the class that does not have some shirks is an exceptional one. When a task is given to be memorized, there is something definite and tangible about it. The pupil and the parent know just what is expected, and as it only involves memorizing, the former knows how to do it, and when it is done, and the teacher, too, knows just what to exact. This explains the popularity and tenacity of memoriter methods in spite of pedagogic literature and normal schools. On the other hand, to assign work of such a kind that the pupil may study hours on it without apparent results, and may finally come to the recitation without any pound of flesh that the teacher can extort; where, in other words, those who have studied long and faithfully are apparently no better off than those who have not studied at all, is a benefit too indefinite to arouse a lazy intellect. As a defence, however, I would say that, holding the memoriter method in geometry to be almost worthless, I regard no study at all as little more so. To quote a witty Irish proverb, one might as well sit idle as work idle. But a skilful teacher will find ways largely to overcome this difficulty; for example, by requiring the laggards to devote to this study a fixed period of the session, or to report the time given to it daily, and how they have applied themselves (of course, avoiding all rewards and punishments that would cause deception), and by reciting to them instances of long and persistent application that have won success.

Here let me answer a question often asked me by those who seem to think a negative answer a complete surrender: do all the pupils in the class do the work equally well and receive an equal benefit? No, neither do they all do the work equally well by the prevailing method, though as to the benefit in the latter case, there can't be much difference. I am not so inexperienced as not to know that any method that aims to secure equal results to every pupil will accomplish it more by repressing the bright and ambitious than by advancing the dull and lazy. What I do claim is, that by the reformed method every one, even the poorest, will get immeasurably more permanent good from the study than he now does. For by a teacher who has any tact, every one can be inspired to do some original work, and where they fail of the final results, they at least will have attained a position to understand quickly the argument of those who have succeeded, and in listening to these explanations they will be acquiring the habit of close attention, in order to be able to reproduce them. This itself will be found to necessitate considerable original thinking, and is the only kind of outside help that should be allowed, save such hints and suggestions as the teacher himself may think best to make from time to time. If it seems inconsistent to admit that any hints or suggestions whatever should be made, let me say that the scholars are not supposed to be fully educated mathematicians, acquainted with all the many devices and circuitous ways by which mathematical results are deduced, and if the demonstration of a proposition depends on the drawing of some secondary line that would not be likely to suggest itself to the pupil, except by too long and discouraging an effort, then it will be better for the teacher at the proper time to suggest it. I say proper time, for the next best thing to having the pupil work out the proof for himself is to get his mind in a state of receptivity for what he is told, and this is always after he has made some little effort of his own; and this, too, is what the use of the common text-books in geometry thwarts. With their ever-ready information they leave the pupil no chance to use what he knows, or to get up an appetite for what he needs to learn. These hints and suggestions, and even explanations, will be chiefly needed in developing definitions, and in the nomenclature where the relations between names and ideas are more or less arbitrary. For example, there is no possible hope that a pupil would ever discover for himself what dividing a line in extreme and mean ratio is, or what an isosceles or a scalene triangle is.

I do not claim that pupils as a whole will get over plane geometry as rapidly as by memorizing it, certainly not at first; neither will they forget it as rapidly, and they will have the power of quick recovery. What I claim is, that it is the right method independent of quantity. I claim, furthermore, that it is in accord with the accepted principles of pedagogics, as enunciated by every prominent writer on the subject from Quintilian to the present, and to which every one deserving the name of teacher gives his assent. If it is not in accord with the universal practice of teachers, it is because the memory is the intellectual path of least resistance, and therefore there is a constant temptation to make it do the most of the mental work.

I claim, too, that it is in exact accord with nature's method by which the race has made intellectual progress. Nowhere has nature written out on the faces of her mysterious productions, as the sun, the rocks, the plants, and animals, a clear and exact explanation of their character, origin, and meaning. She sets her problems and theorems and lets man find out the explanation for himself.

It is, lastly, in exact accord with the method of the Great Teacher of teachers, who in no way manifests his divine character more clearly than by his wonderful knowledge of the science of teaching. Of all his parables, Christ explained to his disciples the meaning of only one, and not even that one till they asked for it, and were, therefore, in a receptive mood to profit by it. He, too, set the problem, and left the world to discipline itself by thinking it out.

CHAPTER II.

ILLUSTRATIONS OF ACTUAL CLASS WORK.

To illustrate the views set forth in the preceding pages, and also to suggest to teachers who may wish to profit by it, how they may be carried out, a few scenes from actual class work may be serviceable. These scenes, founded on fact, as the novelist would say, will be taken from such lessons as require the most explanation and questioning on the part of the teacher; but the dialogue will be shortened wherever it can without obscuring the meaning, by direct hints and suggestions.

RECITATION I.

Let us suppose, then, the class called for its first recitation, for which I would suggest no lesson be previously assigned. Opening the books at the beginning there will be found, instead of the usual definitions, questions calling for those definitions.

The object of this is to utilize such knowledge as the pupils have, and to cultivate exactness and facility in generalizing and defining. Where the question calls for a definition that does not accord with the pupil's previous knowledge and experience, the teacher must, sooner or later, come to his assistance. This will be the case in the very first question.

Teacher. Class, what is a solid? (*Several hands go up.*) John.

John. A solid is a kind of matter where the molecules cling so firmly together that they do not easily change their relative positions.

T. Where did you learn that definition?

J. In studying natural philosophy.

T. But we are studying geometry now; do you think the word "solid" would have the same meaning as in physics?

J. I should think it would.

T. Give an example of a solid.

J. That block of wood on the desk.

T. Well, suppose I could remove this block, leaving a perfect vacuum in its place, just its size and shape, what would you call that?

J. I would call it nothing.

T. Call it nothing, or *a* nothing, John?

J. (*hesitating a little*). I would call it *a* nothing.

T. Would not several of those spaces placed one on the other reach to the ceiling?

J. Yes, sir.

T. Is n't the distance from the floor to the ceiling something?

J. Yes, sir.

T. Then according to your reasoning, several nothings equal something?

J. (*with a puzzled look*). No, sir.

T. Henry, what were we investigating in physics when we used the term "solid"?

Henry. The properties and conditions of matter.

T. What are we to investigate in geometry? (*Henry hesitates.*) Can any one tell me? (*Several hands are raised.*) Mary.

Mary. Lines and surfaces and — solids.

T. But what is there about lines and solids that can be investigated?

M. I do not quite understand.

T. When we were studying physics, Mary, we saw that matter presented to us for investigation certain properties such as hardness, impenetrability, etc.; now has a line any property that distinguishes it in your mind?

M. (*hesitatingly*). Length.

T. Yes. What properties has a surface?

M. (*more confidently*). Length and breadth.

T. Yes. A solid?

M. (*positively*). Length, breadth, and thickness.

T. Can any one in the class give us a general term that embraces the ideas suggested by the words "length, breadth, and thickness"? (*After more or less effort the word "dimensions" is given.*)

The principal subject in geometry is magnitude, or as it may be called, the science of dimensions. Now, William, in a science that treats of matter, the term "solid" signifies a kind of matter, but what should it signify in a science that treats of measurements or dimensions?

W. It should signify something that has measurements or dimensions.

T. Would not the space it now occupies, if the block of wood were taken away, have dimensions?

W. Yes, sir.

T. Then in a geometric sense, what would you call that space?

W. A solid?

The teacher should now see that the term "solid," in its geometric sense, is clearly understood by every member of the class, and especially that they comprehend that the block of wood is not a geometrical solid because it contains a certain kind of matter, but only because it occupies space; that the space only is the true geometrical solid; that space, or any limited portion of it, is a geometrical solid, and that the presence or absence of matter does not make it more or less so. It is also well to incorporate into the definition the fact that it has three dimensions at right angles to each other.

T. Helen, how is a solid bounded?

Helen. By sides.

T. But how about the ends, and the top, and the bottom? Don't they have boundaries?

H. Yes, sir; I meant those too, when I said sides.

T. I see; you meant to give it a technical signification, as we have done with the word "solid." Well, we will have to do that with some word, but can't we find one nearer to our wants than "side"? The objection to the word "side" is that we use it in too many senses. It is sometimes used in the sense we are discussing, but it sometimes signifies a part of the object, as, the sides of the box are made of boards an inch thick. Furthermore, it is sometimes used in distinction from the ends, the top, and the bottom; and sometimes means only a line, as the sides of a square. Now can't some one suggest a word better adapted to our need? (*After some little thinking the word "surface" is suggested.*) That is the word adopted by mathematicians. Does it include any part of the solid, Albert?

Albert. No, sir.

T. And now, Albert, what is a surface?

A. The outside of anything.

T. I fear, Albert, you do not yet clearly comprehend what a geometrical solid is.

A. I think I do, sir.

T. Perhaps so, but when you say "the outside of anything," I am afraid you are thinking of boxes, stones, and such things. "Anything" is not a geometrical solid. Will you please correct your definition?

A. A surface is the outside of a geometrical solid.

T. Hereafter we will drop the adjective "geometrical," and when we use the word "solid" in our work in geometry, we will understand it to mean a geometrical solid. I do not, however, like the word "outside"; it seems to imply an inside where there can be no surfaces. We have just alluded to a use of surfaces; can't you form a definition from that?

A. Surfaces are the boundaries of solids.

T. Very good. Do they take in any of the solid of which they are the boundaries?

A. No, sir.

T. How many dimensions have they at right angles to each other?

A. Two.

T. Please affix that to your definition.

A. Surfaces are the boundaries of solids, and have two dimensions at right angles to each other.

The teacher will do well to explain here that a secondary or incidental use of certain surfaces is as standards for measuring areas.

T. Hattie, what is a plane surface?

Hattie. One that is straight every way.

T. That is true, but perhaps it can be worded better. What is the surface of this black-board?

H. A plane surface.

T. Supposing I marked two points in it and joined them by a straight line, where would that line lie?

H. I do not understand.

T. Well, suppose I take two points in the surface of this globe and join them by a straight line, would that line lie anywhere in the surface of the globe?

H. No, sir.

T. How would it be with the line joining the two points in the surface of the black-board?

H. It would lie in the surface of the board.

T. Wholly or partially?

H. Wholly.

T. How would it be with a straight line joining any two points in the surface of the board?

H. It would be the same.

T. Can you define a plane surface by that test?

H. (*after some effort with the proper wording*). A plane surface is such that a straight line connecting any two points in it lies wholly in the surface.

By this time the recitation period will probably be near its end. It will be well for the teacher, before dismissing the class, to give the pupils a few suggestions on the need of great care in defining, and on the necessity of first grasping the fundamental idea. As their next lesson he should ask them to carefully review the definitions already discussed, and should assign a definite advance, but on no account should they be allowed to write out a single answer or to ask help of others.

RECITATION II.

At the next recitation, after the review with necessary criticisms has been completed, the advance begins.

Teacher. Thomas, what is a curved surface?

Thomas. I could n't think of a good definition for a curved surface, unless you may say that it is such that a straight line connecting any two points in it does not lie in the surface.

T. In general that would be true, but it is faulty in one respect. What would you call the surface of this pencil?

Thos. A curved surface.

T. But if I should take two points along one side (*illustrating*) and connect them by a straight line, where would it lie?

Thos. Wholly in the surface.

T. Then your definition claims too much.

Going through the class, probably no better results will be found.

Teacher. Your difficulty, I think, has arisen from a fact that you have failed to observe; that the term "curved surface" does not have a single, definite meaning, and is therefore hard to define. It is applied to a great variety of surfaces. For example, the surface of that globe, and of this pencil, and of that stone, and of that spiral tube; what kind of surfaces are they?

Several voices. Curved surfaces.

T. Yes, and yet they are all different. Now there is a certain definite surface that we call a plane surface. What do we call all others?

Several voices. Curved surfaces.

T. Thomas, can you define a curved surface by that simple test?

Thos. A curved surface is one that is not a plane surface.

T. We will accept that, although, as a general thing, negative definitions are to be avoided. But the common idea in all curved surfaces seems to be that they are such that straight lines extending at most in more than one direction cannot lie wholly in them, and that is what the definition just given by Thomas means. Anna, how are surfaces bounded?

Anna. Surfaces are bounded by lines.

T. What, then, is a line?

A. It is the boundary of a surface.

T. Repeat the definition we have agreed on for a solid, Anna. (*Anna does so*) Now repeat the one for a surface. (*She does so.*) Now make your definition for a line as complete, Anna.

A. A line is the boundary of a surface, and has only one dimension.

T. Can you mention any incidental uses of lines?

A. (*thinking a moment*). Certain lines may be taken as standards to measure distances by.

T. Yes, and another important use of lines is to indicate direction. (*After seeing that this is understood.*) I wish to-morrow in the review you would incorporate this into the definition. James, does a line include any part of the surface?

James. No, sir.

T. (*drawing a chalk mark on the board*). Is that a line?

J. Yes, sir.

T. (*measuring its width*). This takes in some of the surface.

J. (*hastily*). No, sir; that is n't a line. It only represents a line.

T. (*after seeing that this is made clear*). I think you all understand it. But as the expression, "the representative of a line," is rather clumsy for constant use, custom permits us to use the word "line" alone in this sense, and you will not, I think, be led into error by it. Now, can any one tell me what a straight line is? (*Several hands are raised.*) Katie.

Katie. A straight line is one that always has the same direction.

T. What kind of a line would the edge of this ruler represent?

K. A straight line.

T. (*turning the ruler in different directions*). By your definition it is n't, for you can see for yourself it does n't always have the same direction. (*Katie looks puzzled.*) How can you answer that, Katie?

K. My definition is wrong.

T. Correct it then. Your definition certainly implies that a straight line always has the same direction in space. Now, what is wrong about it?

K. No, I did n't mean that. It always has the same direction in all its parts whichever way you turn it.

T. Now you have hit upon the right idea. Will you, then, define a straight line?

K. A straight line is one that always has the same direction in all its parts.

T. I do not quite understand why you use the word "always." Do you mean that if the edge of this ruler got bent next week, that would forbid its being called straight now?

K. No, sir (*thinking a moment*). "Always" should be left out.

T. Can any one suggest a better expression than "in all its parts"? (*Probably some scholar will suggest the word "throughout." If not, the teacher may do so, after the class has thought for a few moments.*)

T. Julia, what is a curved line?

Julia. One that changes its direction.

T. (*again turning the ruler*). Then the line represented by the edge of this ruler is a curved line, for you see it keeps changing its direction.

J. (*trying again*). It is one that is bent.

T. (*drawing a line thus* , *or bending in the same way some straight object*). Then this is a curved line.

J. No, sir; not exactly.

T. It is bent, is it not?

J. Yes, sir.

T. Then it fulfils your condition.

J. It should be bent, everywhere.

T. I think this would be bent wherever I might put it.

J. I mean at every place in it.

T. What do you you mean by "every place in it"?

J. Every point in it.

T. Julia, you know what a curved line is; your difficulty is you do not express your knowledge clearly and correctly. Please repeat the definition of a straight line. (*Julia does so.*) Now in drawing a straight line (*drawing one slowly*) what do I have to do as to direction?

J. You have to keep the same direction throughout.

T. In drawing a curved line (*drawing one slowly*) what do I have to do?

J. Keep changing the direction.

T. At certain points? (*Illustrating.*) That would not make a curve, would it?

J. No, sir; at every point.

T. Right. Now define a curved line.

J. A curved line is one that keeps changing its direction at every poin

T. Would not "changes" be as good as "keeps changing"? Repeat with that alteration and see. (*Julia does so, and all agree that it is the better.*)

In this discussion, the teacher should not forget that the conditions are very different from what they were in the discussion of surfaces and solids. The pupils did not know what geometrical surfaces and solids are, but they do know what lines, straight and curved, are, and the teacher should be in no haste to assist them.

T. Albert, what is a point?

Albert. A point is something that has neither length, breadth, nor thickness.

T. If it had neither length, breadth, nor thickness, I would call it a nothing. Would n't you?

A. Yes, sir.

T. Then according to your definition a point is something that is a nothing. Please think what a point is used for. When you think of a point in a line or surface, what is the chief idea you associate with it?

A. Place.

T. The word "place" seems to suggest area, capacity. There is a word to which this objection cannot be urged. Can any one suggest it? (*Sevreal hands are raised.*) Henry.

Henry. Position.

T. Will you, then, define a point, Henry?

H. A point has position only.

T. Which do you think would be the better verb to use "has" or "indicates"? (*After a moment's thought Henry decides in favor of indicates.*) Is "only" necessary?

H. (*thinking*). No, sir.

T. But what about dimensions?

H. It has none.

T. Please incorporate that into the definition.

H. A point indicates position, and has no dimensions.

T. If a point should be moved, Helen, what would be produced?

Helen. A line.

T. How then may a line be generated?

H. By moving a point.

As the recitation period is at an end, the teacher will assign the next lesson. Perhaps it may be well for him to again caution the pupils against receiving aid, and especially against writing out the definitions, for these definitions should be a part of their mental armament, and not of a written record.

Let me say here that I have omitted in these illustrations, directions given to the class as to how by letters and figures, surfaces, lines, etc., may be designated, as this is a matter of simple instruction on the part of the teacher.

Instead of taking the lessons in order, let us next pass to the one in which angles are discussed.

RECITATION III.

T. John, what is an angle ?

John. The space between two lines.

T. (*drawing two parallel lines on the board and pointing to the space between*). This is an angle ?

J. No, sir ; I meant two lines meeting at a point.

T. Very well, John ; you ought to say at first exactly what you mean. You have been studying geometry long enough to know that you ought not to say one thing when you mean another. Now will you state just what you mean?

J. An angle is the space between two lines meeting at a point.

T. Will you go to the board and draw one? (*John does so.*) Now I think I understand you. The angle is the space between those two lines.

J. Yes, sir.

T. Lengthen the lines, please. (*John does so.*) That makes the angle larger, does it?

J. No, sir; to make the angle —

T. Wait, John; you said that the space between two lines meeting at a point is an angle. Now does not lengthening the lines make the space between them larger?

J. Yes, sir; but —

T. A moment, John; do you confess that you have not yet said what you mean?

J. Yes, sir.

T. Well, then, will you draw an angle larger than the one already drawn? (*John does so.*) That is correct, and seems to show that you know what angles are. Now can you give it a correct expression?

J. An angle is — I don't think I can define it.

T. Draw two lines on the board as nearly as you can in the same direction. (*John does so.*) Could those two lines form an angle?

J. No, sir.

T. Why?

J. Because they would not meet.

T. Why will they not meet?

J. Because they have the same direction.

T. What must be true of their directions that they may form an angle?

J. They must have different directions.

T. What, then, is an angle solely determined by?

(*As John is still uncertain, William is called on.*)

William. By the difference in the direction of the lines.

T. What, then, is an angle?

W. An angle is the difference in the direction of two lines meeting at a point.

T. Please write it on the board. (*William does so, and also at the teacher's request, two other phrases so that they will read as follows:—*

The difference in the direction of two lines;
The difference in the directions of two lines;
The difference in direction of two lines.)

Which of these expressions, class, is the best, and is either of them incorrect? (*If no satisfactory answer is given by the pupils, they should be asked to think it over, and report at the next recitation. The teacher may further suggest that instead of the phrase "meeting at a point," "originating from the same point" be used.*

T. Mary, what is a right angle?

Mary. It is a square angle.

T. Please draw one. (*Mary does so, and at the teacher's request an acute angle, and an obtuse angle also, as follows :—*)

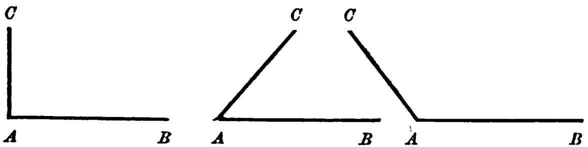


FIG. 1. FIG. 2. FIG. 3.

Now, Mary, look carefully at those figures and tell me what you go by when you call the first one a right angle.

M. (*thoughtfully*). By the position of the line A C.

T. What is its position ?

M. It is perpendicular to the line A B.

T. Are the lines always so related in forming a right angle ?

M. Yes, sir.

T. Then why is n't that a good way to define it ? Let me ask you again ; what is a right angle ?

M. A right angle is one formed by a straight line and a perpendicular to it.

T. Which is the straight line, and which the perpendicular ?

M. A B is the straight line, and A C the perpendicular.

T. But is not A C a straight line ?

M. Yes, sir.

T. And is not A B perpendicular to it ?

M. Yes, sir.

T. Then how could you express their relation without giving the impression that one of them is more important than the other?

M. They are perpendicular to each other.

T. Now define a right angle.

M. A right angle is one formed by two lines perpendicular to each other.

T. Anna, what is an acute angle?

Anna. An acute angle is one formed by lines oblique to each other?

T. That definition would include an obtuse angle, too, would it not?

A. (*thinking a moment*). Yes, sir.

T. How does an acute angle compare in size with a right angle?

A. It is smaller.

T. Always?

A. Yes, sir.

T. Why, then, would n't that be a good way to define it?

A. An acute angle is an angle smaller than a right angle.

T. James, what is an obtuse angle?

James. An obtuse angle is an angle larger than a right angle.

The teacher should explain without further questioning the straight and the reflex angle, as the pupils probably know nothing about such angles.

Teacher. Thomas, to return to perpendicular lines, when is a line perpendicular to another?

Thomas. When it forms a right angle.

T. But we have defined a right angle as one that is formed by two lines perpendicular to each other. That kind of defining reminds me some what of a game of lawn tennis. Do you think it a good way to define?

Thos. No, sir.

T. Well, you may sit. Now, class, the question is this: which has precedence, the perpendicular or the right angle? In other words, is the perpendicular the primary idea and right angle a name given to the angle formed by it, or is the right angle the primary idea and perpendicular the name given to the position of each of its sides? If the perpendicular has the precedence, then we ought to define it without using the term "right angle," for that is a name given to something that the perpendicular produces. If the right angle has precedence, then we ought to define it without mentioning the perpendicular, for the same reason. Now which is it?

A discussion will probably show the class divided, and the teacher will have to be the final arbiter. For my own part, I prefer to consider the perpendicular the prior idea, and to define it as a straight line so related to another as to form with it equal adjacent angles. An oblique line, of course, would be defined as one so related to another as to form unequal adjacent angles.

Everything pertaining to the understanding of angles should be made clear to the pupils, as this is the open sesame to geometry. For this reason, I have always deemed it advisable to add a few supplementary axioms pertaining to angles.

Can it be doubted that the development of definitions in some such manner as I have indicated is incomparably better than memorizing them as given in the standard text-books? One great objection to the use of such text-books is that no allowance is or can be made for what the pupil knows. This, in my opinion, is a fatal defect, for it is of infinitely greater value to acquire facility in using the knowledge we have than to rememorize it. Indeed, should not this be the chief aim in education? Furthermore, in memorizing, the pupils take the definitions only in the rough, and fail to cultivate that sense of clear comprehension, and that power of correctly defining, that are together the very essence of logic: a rare acquisition, let me add, even among scholars.

[illegible]RATION IV.

[illegible] pass on to the first lesson in [illegible].

[illegible], what is proposition first?

[illegible] one straight line meets an-[illegible] adjacent angles, the sum of [illegible] two right angles.

[illegible] to be done?

[illegible] the figure

[illegible]

T. Why did you specify one rather than the other? If you should leave out "by drawing a figure," do you think the answer would be less clear?

O. (*thinking*). No, sir.

T. Then how would your answer stand?

O. To show what the theorem means.

T. Can any one state that better? (*Several hands.*) Julia.

Julia. To illustrate the theorem.

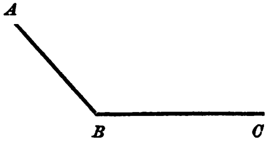
T. That is a great deal better. It is more concise and states the fact more correctly. Now, Julia, to illustrate, as has been said, we almost always draw a figure. Where do you find your data for drawing the figure?

J. In the theorem.

T. In this theorem what are they? Read them.

J. If one straight line meets another so as to form adjacent angles, the sum —

T. Just the data, Julia, and no more. (*Julia repeats them correctly.*) Now you may illustrate. (*Julia going to the board draws the following figure:* —)



Julia. The line A B meets the line A C —

T. (*waiting a few moments*). Are you sure you have drawn all the conditions, Julia? Please read

RECITATION IV.

Let us now pass on to the first lesson in demonstration.

Teacher. Ora, what is proposition first?

Ora (*reading*). If one straight line meets another so as to form adjacent angles, the sum of these angles is equal to two right angles.

T. What is the first thing to be done?

O. To draw a figure.

T. (*stepping to the board and drawing the figure* 6). What next?

O. I did n't mean that kind of a figure.

T. There was nothing in your answer to imply that it made any difference as to the kind of figure. Well, what kind did you mean?

O. One to show what the proposition means.

T. Which is the more important, to draw the figure or to show what the theorem means?

O. To show what it means.

T. The form of your answer seemed to make drawing the figure the more important. (*Teacher explains why.*) Will you please restate what you consider the first thing to be done?

O. To show what the theorem means by drawing a figure.

T. Is drawing the figure the only thing necessary to show what the theorem means?

O. No, sir; it has to be explained.

T. Why did you specify one rather than the other? If you should leave out "by drawing a figure," do you think the answer would be less clear?

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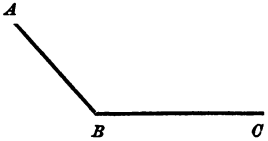

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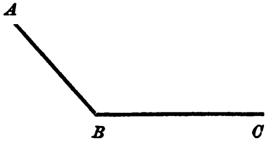

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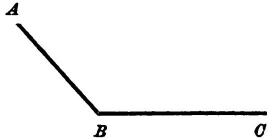
J. If one straight line meets another so as to form adjacent angles, the sum —

T. Just the data, Julia, and no more. (*Julia repeats them correctly.*) Now you may illustrate. (*Julia going to the board draws the following figure:—*)



Julia. The line A B meets the line A C —

T. (*waiting a few moments*). Are you sure you have drawn all the conditions, Julia? Please read

RECITATION IV.

Let us now pass on to the first lesson in demonstration.

Teacher. Ora, what is proposition first?

Ora (*reading*). If one straight line meets another so as to form adjacent angles, the sum of these angles is equal to two right angles.

T. What is the first thing to be done?

O. To draw a figure.

T. (*stepping to the board and drawing the figure* 6). What next?

O. I did n't mean that kind of a figure.

T. There was nothing in your answer to imply that it made any difference as to the kind of figure. Well, what kind did you mean?

O. One to show what the proposition means.

T. Which is the more important, to draw the figure or to show what the theorem means?

O. To show what it means.

T. The form of your answer seemed to make drawing the figure the more important. (*Teacher explains why.*) Will you please restate what you consider the first thing to be done?

O. To show what the theorem means by drawing a figure.

T. Is drawing the figure the only thing necessary to show what the theorem means?

O. No, sir; it has to be explained.

T. Why did you specify one rather than the other? If you should leave out "by drawing a figure," do you think the answer would be less clear?

O. (*thinking*). No, sir.

T. Then how would your answer stand?

O. To show what the theorem means.

T. Can any one state that better? (*Several hands.*) Julia.

Julia. To illustrate the theorem.

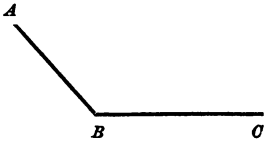
T. That is a great deal better. It is more concise and states the fact more correctly. Now, Julia, to illustrate, as has been said, we almost always draw a figure. Where do you find your data for drawing the figure?

J. In the theorem.

T. In this theorem what are they? Read them.

J. If one straight line meets another so as to form adjacent angles, the sum —

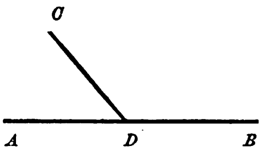
T. Just the data, Julia, and no more. (*Julia repeats them correctly.*) Now you may illustrate. (*Julia going to the board draws the following figure: —*)



Julia. The line A B meets the line A C —

T. (*waiting a few moments*). Are you sure you have drawn all the conditions, Julia? Please read

again. (*Julia reads the theorem again, and studies her figure. If she does not yet see the defect, the teacher should keep her thinking till she does see it, or till a number of hands are up, and some one draws the figure correctly, as follows:* —)



T. Frank, explain please?

Frank. The line C D meets the line A B at D.

T. Well, what then?

F. Then the angles A D C and C D B are equal to two right angles.

T. I don't think that the theorem says that the angles A D C and C D B are equal to two right angles. Please read it.

F. (*after re-reading*). The sum of the angles A D C and C D B is equal to two right angles.

T. That is correct.

The teacher may now show how the first part of the explanation should be postulated as, Let the line C D meet the line A B so as to form the adjacent angles, etc.

T. What is the next thing to be done, class?

Several. To prove it.

T. Right. Albert, what is the assertion made in the theorem?

Albert. That the sum of the angles A D C and C D B is equal to two right angles.

T. You may prove it.

A. I don't think I can.

T. You may sit. Now, class, do you think it likely that this theorem has been given you before you have had sufficient geometrical facts laid before you to prove it by?

Several. No, sir.

T. Certainly not, each one of you has or ought to know enough to prove this, if you can only make use of that knowledge. Now, what do you know? (*Several hands.*) Hattie.

Hattie. The definitions.

T. What else? (*After a little thinking, several hands.*) John.

John. The axioms.

T. Correct. What particular definitions do we need in this case? (*Several hands.*) Mary.

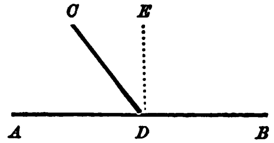
Mary. The definitions of angles.

T. Let us look them over and see what use we can make of them. (*After a few minutes spent in reviewing these definitions, a hand is raised.*) Anna.

Anna. If C D was perpendicular, the two angles would be right angles.

T. You may draw a line in that position, and make it a dotted line to distinguish it from the primary lines specified in the theorem. (*Anna does so.*) Now what do you claim?

A. That the angles A D E and E D B are two right angles.



T. We will have to grant that. But we want to prove something about the angles A D C and C D B. What is it?

A. That they are equal —

T. Careful, Anna, look at the theorem.

A. That their sum is equal to two right angles. (*Several hands go up eagerly, and Anna goes on.*) The sum of the angles A D C and C D B is equal to the angles A D E and E D B.

T. You mean the sum of the angles A D E and E D B, do you not?

A. Yes, sir.

T. What makes you think so?

A. Because they are the same size.

T. That 's merely a repetition, Anna. What do the rest of you say to this? (*They all agree with Anna, and can't see what more the teacher can ask.*) Now, scholars, the difficulty is this: you have some reason that convinces you that the sum of the angles A D C and C D B is equal to the sum of the two right angles A D E and E D B, but it is so simple you don't think it necessary to state it. But in demonstrating a proposition, no step, however simple, needed to make the proof clear, should be omitted. Now what is the step that you are leaping over? (*A hand is raised.*) Henry.

Henry. A D C is smaller —

T. What do you mean by A D C?

H. The angle A D C is smaller than the right angle A D E, and the angle C D B is larger than the right angle E D B.

T. Hence what? Finish, Henry, please.

H. Hence A D C and —

T. Be careful of your statement, Henry.

H. Hence the sum of the angles A D C and C D B is equal to the angles —

T. Sum of the angles, Henry.

H. (*continuing*). Is equal to the sum of the two right angles A D E and E D B.

T. Or to reason in the same way, Lowell and Lawrence are both the same distance from Boston, about twenty-five miles. San Francisco is farther from Boston than Lowell, and Stoneham is nearer than Lawrence. Hence the sum of the distances from Boston to San Francisco and from Boston to Stoneham is equal to the sum of the distances from Boston to Lowell and from Boston to Lawrence. Now is that so?

H. No, sir.

T. It is in exact accord with your reasoning. (*A hand is raised.*) Mary.

Mary. The angle A D C is as much smaller than the right angle A D E, as the angle C D B is greater than the right angle E D B. (*Hesitates.*)

T. Go on, Mary.

M. Hence the sum of the angles A D C and C D B equals the sum of the two right angles A D E and E D B.

T. Good, but what is the inference, Mary? (*If the scholars do not understand this, the teacher will explain.*)

M. (*finishing*). Therefore if one straight line meets another, etc.

T. That is quite good, but I think we can improve it a little as to clearness. Tell me, some one, what the perpendicular D E forms.

John. Two right angles.

T. Right. What does it do to the angle C D B?

Julia. It divides it into two parts.

T. Do you know anything definite about either of these parts?

J. One of them E D B is a right angle.

T. Well, what about the other?

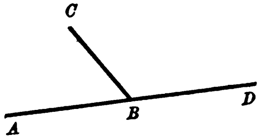
William. The other added to A D C forms the right angle A D E.

All agree that this is clearer. As far as time will permit, certain pupils will be asked to go through the whole demonstration *in toto*, the teacher making suggestions regarding the use of such words as "let," "then," "hence," "therefore," etc. The corollaries will also be explained by the pupils. Attention will also be called to the matter of illustration, and it will be impressed upon the pupils that after one or two lessons they will be expected to go to the board and illustrate any proposition assigned them without further remark. For the following lesson, Proposition I. and its corollaries, and perhaps certain definitions, will be assigned as a review, and as an advance, Proposition II., which they will be asked to illustrate and demonstrate if possible.

Let me add that this minute and detailed method of questioning will not be long needed if the teacher has tact and is persistent in the outset.

RECITATION V.

In the next recitation, after the review, Proposition II. will be taken up. It will be found that all, or nearly all, can properly illustrate it, and by drawing, as they probably will, the lines from A to B, from D to B, and from C to B, they will get rid of one difficulty



that pupils always have with this proposition, in using the regular text-books where the figure is drawn; namely, the difficulty of A D appearing from the start to be a straight line.

Some of the pupils will even go further, and have what seemed to them a demonstration. These the teacher will patiently hear, pointing out where their reasoning does not connect. If any have it right, it will be an evidence that they are receiving assistance from some source, and a private interview and admonition should follow at the first convenient moment.

But to return to the recitation, the teacher should now explain to the class the indirect method of proof, making it clear by contrasting it with the direct method as illustrated in Proposition I. To show how a demonstration is worked out by this method, do not take Proposition II., but make up some simple theorem like the following: —

Teacher. What is a perpendicular, Ruth?

Ruth. A perpendicular is a line so situated with reference to another that the adjacent angles formed by their meeting are equal.

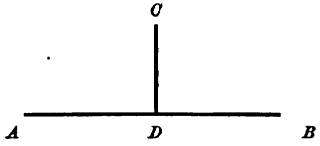
T. Then if you know a line is perpendicular to another, what can you assert of the adjacent angles?

R. That they are equal.

T. True. Now suppose the converse, that is, that you know the adjacent angles are equal, what can you assert about the line forming them?

R. That it is a perpendicular.

T. Well, if called upon to prove it, how would you do it? (*Ruth hesitates.*) You may sit, Ruth, and I will show you how I would do it. (*Drawing figure.*) Let the line C D meet the line A B so that the angles A D C and C D B are equal, then C D is perpendicular to A B. Now, class, to pause for a moment, let me say that if called to prove this directly I might find it difficult to make my argument clear. But one thing we do know; we know that there can be a perpendicular to the line



A B, at the point D, and if C D is not that perpendicular some other line can be. We can be sure of that, can't we? (*Several voices: Yes, sir.*) Then if I can prove that no other line by the conditions given me in the theorem can be perpendicular, must it not follow that C D is, just as certainly as if I had proved it directly? (*Several voices: Yes, sir.*)

T. Well, then, I will suppose for the sake of argument that C D is not perpendicular but that E D is. Now as long as it is not positively known that C D is perpendicular, I have a right to make such a supposition, but not after it is once proven. Herbert, have I a right to make a supposition that the angle A D C is not equal to the angle C D B?

Herbert. No, sir.

T. Why not?

H. (*after a moment's thought*). Because the theorem states that they are equal.

T. But the theorem states that C D is perpendicular. What is the difference? (*Herbert is puzzled.*) Let us write out the theorem, and then we can study it better. Who can state it for me?

After several attempts by the members of the class, the following form is adopted:—

If the adjacent angles formed by a straight line meeting another are equal, the line is a perpendicular.

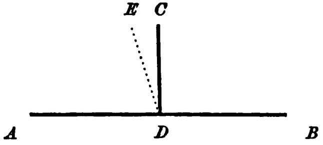
T. Now why can I not suppose the angles to be unequal as well as C D not to be perpendicular? (*Hands are raised.*) Mary.

Mary. Because the theorem gives the angles equal, but it has not yet been proved that C D is perpendicular.

T. In other words, it is not merely *stated* in the theorem that the angles are equal, but the theorem is based on the condition that they are, so that no change in that respect is allowable.

After seeing that all the class clearly understand this, and that the supposition must always be made on the unproven assertion, and contrary to it, the teacher continues: —

T. As we have interrupted the discussion to explain the method, I will commence again.



Let the angles A D C and C D B be equal, then the line C D is perpendicular to A B. Suppose that C D is not perpendicular, but that E D is; then the angles A D E and E D B are equal, for a perpendicular forms equal adjacent angles. Hence while the angles A D C and C D B are equal, the angle A D E, which is smaller than the angle A D C, is equal to the angle E D B, which is larger than C D B; an evident absurdity. Now, scholars, where is the error that led me to this absurdity? Have I not reasoned correctly?

Several voices. Yes, sir.

T. Then who can tell me what error led me to this absurdity? (*Several hands.*) John.

John. The error was in the supposition.

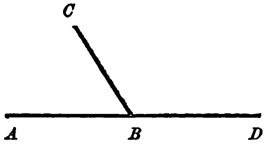
T. What was the supposition?

J. That C D was not perpendicular.

T. Very well, if that is an error, what must be true?

J. That C D is perpendicular.

T. Yes, and demonstrated to be so just as surely as if it had been done directly. Now let us return to Proposition II. William, please give the theorem again. (*William does so.*) Who can demonstrate it? (*After some little thinking by all the class, Julia raises her hand.*) Well, Julia.

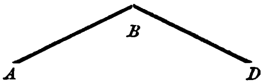


Julia. I could prove it if C B was perpendicular.

T. Perpendicular to what?

J. To A D.

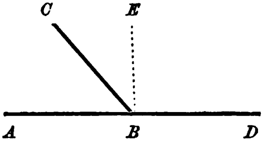
T. But A D is made up of two lines, and if it is not straight, could you draw a perpendicular to it. For example (*drawing a line thus*):



could you draw a line at B perpendicular to this A D?

J. No, sir; but you can draw it perpendicular to A B.

T. Very well, you may do so; make it a dotted line, please, so that we can distinguish it from the construction lines already drawn. (*Julia does so.*)



Now what is your proof?

J. The angles A B E and E B D are two right angles. (*Hesitates.*)

T. Why are they?

J. Because a perpendicular forms two right angles.

T. What authority have you for that?

J. A perpendicular forms two right angles.

T. Yes; but what is E B perpendicular to?

J. To A B.

T. Do you know that it is perpendicular to B D?

J. No, sir.

T. Then how do you know that the angle E B D is a right angle?

J. I don't know.

T. Well, Julia, there is a way to prove it. Can any one see how? (*After a little thinking two or three hands are raised.*) Well, Albert.

Albert. The sum of the angles A B C and C B D is equal to two right angles, and the right angle A B E plus the E B D are equal to A B C plus C B D, therefore A B E plus E B D are equal to two right angles.

T. Very good; and since A B E is a right angle what must E B D be?

A. A right angle.

T. That is good as far as you have gone, but we don't want to stop there. We want to prove that A B and B D form one straight line. What were you going to do next, Julia?

Julia. I was going to suppose that A B and B D do not form a straight line.

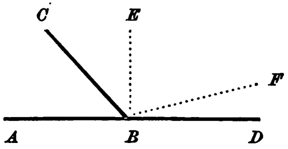
T. Well, what would you do then?

J. I would make them straight.

T. What, by changing them? I do not think that would be allowable.

J. I meant draw one of them straight.

T. Do it as you were intending, Julia. (*Julia draws the line* B F *as follows.*)



Now, continue your argument.

J. The angles A B E and E B F are two right angles, and (*very hesitatingly*) E B F is equal to E B D. (*Stops.*)

T. Have you finished, Julia? I thought you were to prove that A B and B D formed one straight line; I have n't heard you draw any such a conclusion.

J. E B F cannot be equal to E B D, therefore —

T. Don't forget to give us the reason why, Julia.

J. (*hesitating*). Because it is n't as large.

T. Is it self-evident?

J. Yes, sir.

T. Then it ought to be found among the axioms.

J. (*looking over the axioms*). The whole is greater than any of its parts, therefore we have come to an absurdity, and A B and B D must form a straight line.

T. You have proved the statement made in the theorem, Julia, but the questions now arise, have you done it in the most direct way? Have you inserted any unnecessary steps? Have you arranged your successive steps logically? Have you expressed them clearly and concisely? And now to test these points, let me first ask if it is necessary to draw the perpendicular E B. All please examine the figure, and give me your opinion. (*After a little thinking several hands are raised.*) Thomas.

Thomas. I think not, for it could be proved that by the supposition the angles C B D and C B F are equal.

T. Please do so.

Thos. The angles A B C and C B D are equal to two right angles —

T. Please don't you mean the sum of the angles A B C and C B D?

Thos. Yes, sir; the sum of the angles A B C and C B D is equal to two right angles, but if A F is a straight line the sum of the angles A B C and C B F is equal to two right angles. Hence the sum of the angles A B C and C B D equals the sum of the angles A B C and C B F. (*Stops*)

T. I thought you were going to prove that under the circumstances the angle C B D is equal to the angle C B F.

The teacher will hold the attention of the class steadily to this point till Thomas or some one sees that by subtracting the common angle A B C the desired result is obtained, and then he will have Thomas finish the proof. He will also criticise Thomas for omitting to state the reasons for his assertions as to the sums of the angles as above, and will have him insert "from the conditions of the theorem" before one, and after the other, cite the theorem of Proposition I.

T. There is one thing more I want to call especial attention to. Can we not get at the fact that the angles C B D and C B F are equal in another way? According to the conditions of the theorem, what is the relation of the angle C B D to the angle A B C?

Several voices. A supplement.

T. If the supposition is correct, what other supplement has the angle A B C?

Several voices. The angle C B F.

T. Helen, what then follows?

Helen. That the angle C B D equals the angle C B F, for all supplements of the same angle are equal.

All agree that this is the more concise way of putting it. After a few other criticisms on certain forms of expression that the teacher has noted, the remainder of the hour is spent in demonstrating the theorem entire, by as many of the pupils as time permits. Of course there will be many occasions for criticism, and these the teacher must carefully call attention to, and as far as possible have the pupils themselves correct the faults. All care thus taken at first will repay the time it costs, by better results afterwards. The scholars should be required at first to state authorities in their proper connection. Later the more common ones may be omitted.

T. Now, scholars, has there been any principle used in the demonstration of this proposition that you did not know, except the method which I explained to you?

All. No, sir.

T. Do you think that there will be anything required in proving the next proposition that you do not already know?

All. No, sir.

T. Do you think that you will have to use everything that you have hitherto learned?

There is, at first, some doubt about this, but it is finally decided in the negative.

T. Very well, then, after illustrating the theorem, your judgment will have to be exercised in selecting such facts as bear on the case, and in marshalling them so as to prove the assertion. And again let me caution you against receiving any outside help.

RECITATION VI.

Teacher. Henry, you may demonstrate Proposition I. (*To save time the figure has previously been drawn.*) (*Henry does so.*)

T. Hattie, you may demonstrate Proposition II.

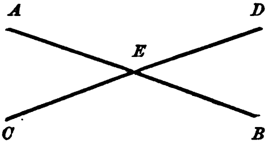
Hattie undertakes it, and on getting about half through gets confused, and John is called upon, who finishes it, though not without some little slips that are duly criticised by the class or the teacher.

T. Helen, you may take Proposition III.

Helen. I don't think I can.

T. Please do not let me hear again that excuse from any one in the class. You can at least state the theorem. You ought to go further and illustrate it. When I call upon you I want you to rise and go just as far as you can, and offer no excuses unless they are called for. Then we will see what is to be done next.

Helen. If two straight lines intersect each other, the vertical angles are equal. (*Drawing *figure.*) Let the lines A B and C D intersect each other at the point E, then the angle A E C will



be equal to the vertical angle D E B. (*Lays down the pointer.*)

T. You made an effort to think out a demonstration, did you not, Helen?

H. Yes, sir.

T. Tell me as nearly as you can estimate how long you thought over it, after you had finished the illustration.

H. (*hesitating*). About ten minutes.

T. What made you give it up?

H. I could n't see how it was to be proved.

T. Do you mean that you could n't see through it in ten minutes, or that you could n't see through it at all.

H. That I could n't see through it at all.

Questioning the class, the teacher finds several whose effort has been similar.

T. I hardly think that you do yourself credit, scholars, in concluding that you can't think out a point at all because you can't do it in ten minutes. Ten minutes is hardly time enough to get the brain fairly working. And now, class, I am not surprised that Helen, or that any of you who have devoted to thinking on this proposition no more than ten minutes, should fail to work it out, but I am surprised that you should make no more effort. Why, many an instance might be cited where those who have developed a science, especially that of mathematics, have, without giving up, spent upon a problem more days, weeks, months, or even years, than Helen has minutes. You have all the advantage that could be asked for over those men, for you know that the propositions set you can be demonstrated, and that you have all the facts needed in the demonstration; they studied to find new principles, not knowing whether what they sought was possible. Leibnitz, who shares with Newton the honor of having discovered the differential calculus, the highest and most wonderful of all the branches of mathematics, devoted years of hard study to the problem, and we are told by his biographers, that when he had some especially hard point in hand, he would scarcely leave his desk for months. Newton worked thirty years on his "Chronology of Ancient Kingdoms." Of course I do not expect you to equal these, but I mention them as a contrast to the ten little minutes that sufficed to convince some of you that your task was a hopeless one.

In a book* familiar to past generations of students, but well-nigh unknown to the present, there is one passage that especially impressed itself upon me and is so full of good suggestions that I cannot forbear quoting it. It is as follows:—

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* Watts on the Mind.

> "Nor yet let any student on the other hand frighten himself at every turn, with insurmountable difficulties; nor imagine that the truth is wrapt up in impenetrable darkness. These are formidable spectres which the mind raises sometimes to flatter its own laziness. Those things which in a remote and confused view seem very obscure and perplexed may be approached by gentle and regular steps, and may then unfold and explain themselves at large to the eye. The hardest problems in geometry, and the most intricate schemes or diagrams, may be explicated and understood step by step. Every great mathematician bears a constant witness to this observation."

Let me add in concluding this little lecture that if you put persistent, determined effort into mental work, you will be surprised at what your mind will do for you, that seemed to you at first impossible. Why do you not put forth what is in you? Are you waiting till some one else explains a thing to you so that you can understand it? Do you think that the highest mental effort of which you are capable is to merely understand an explanation? Respect for your own ability ought to make you ashamed to think so. Dig out the explanation. You long ago got your bodies out of your baby-carriages; try to get your minds out also.

The class should now be set to work studying on the proposition. If any claim to have found a demonstration, the teacher should hear them individually and privately. If they have failed, he should simply point out the defect and let them think out the correction. If they have succeeded, they should be commended, and asked to work out some of the additional propositions at the end of the book. If none succeed, which is not probable, the proposition should be continued till the next lesson. It is important that now the scholars should learn that, in the easier propositions at least, they must not look for much assistance. It is not, however, best to delay too long for every scholar to think it out; but when in the judgment of the teacher a fair number have worked it out, some one may be called on to present his demonstration to the class, and attention will be given to the clearness and correctness of his style.

RECITATION VII.

Let our next recitation include Proposition V.

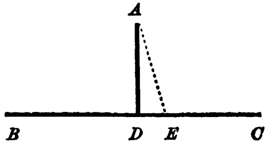
Theorem: From any point outside of a

straight line there can be but one perpendicular drawn to the line.

After the theorem has been stated, the pupils, as will be best at least for the present, should all be requested to draw a figure to illustrate it. For this purpose, the scholars should always bring with them into the class the necessary implements. After it is seen that each has correctly represented the conditions, the figure can be transferred to the board, or the demonstration may go on from the papers.

Teacher. Henry, you may commence.

Henry. Let A be a point outside of the line B C, then only one perpendicular A D can be drawn to B C. Suppose there can be another perpendicular as A E. Since A D is perpendicular to B C, the angle A D C is a right angle. (*Gives authority.*) Since A E is perpendicular to B C, the angle A E C is a right angle. Hence the angle A D C is equal to the angle A E C (*authority*), but this is absurd, for the angle A D C is smaller than the angle A E C. Therefore the sup-



position is wrong, and there can be but one perpendicular drawn from A to the line B C.

T. Well, Henry, let me question you a little. You said that "*Since* A D is perpendicular to B C, the angle A D C is a right angle," which is correct. But you went on to say, "*Since* A E is perpendicular to B C." Now I thought that was only a supposition.

H. It was.

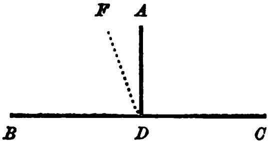
T. But when you say, "Since A E is perpendicular," you practically assert that it is so, just as you did in regard to A D. Is that proper, and how can you avoid the implied assertion?

H. If the line A E is perpendicular.

T. That is better. Again you asserted that the angle A D C is smaller than the angle A E C; how do you know that?

H. (*hesitating*). A D C is a part of A E C.

T. I don't see how you know that. If their vertices were together, thus (*draws*): as A D C and F D C have their vertices at D, I could see that one was a part of the other. But they are not, and I don't see why you should say that one is greater or smaller than the other, unless you guess at it, and guessing, however good, is n't good

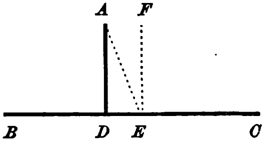


geometry. (*Henry, failing to explain, sits.*) Well, class, Henry has made an argument and reached the results asserted in the theorem; but we find that he has begged one point, claiming what he had n't and could n't prove, and so vitiated his whole argument. Now, it may be what he claimed is true, and I am strongly inclined to believe it, but it won't do to allow guessing. If we once allow guessing, there is no knowing where we shall land. What, then, shall we do? (*After a little thinking, hands begin to go up.*) Julia.

Julia. At E draw a line parallel to A D.

T. Please do so.

J. (*drawing E F*). Then the angles A D C and F E C are equal, for both are right angles. But F E C is only a part of A E C. (*Stops.*)



T. (*waiting a moment*). Have you got through, Julia? I did n't hear you draw the conclusion that there can be but one perpendicular.

J. Therefore from any point outside of a straight line there can be but one perpendicular to the line.

T. How do you get that? Why do you conclude that our supposition is wrong? You have not pointed out any absurdity, as I have heard.

On the contrary, all the statements and conclusions you have given us seem to me correct, and therefore we ought to think the supposition right. Perhaps, Julia, you had better begin all over again, and demonstrate the theorem all over again.

J. (*after giving the theorem*). Let A be a point outside of the line B C, and A D a perpendicular to B C, then there can be no other perpendicular from the point A. Let us suppose that there can be another to the point E, as A E. Then from the point E —

T. Now, Julia, you are skipping. There are some facts about the lines A D and A E that you will find no better place to state.

J. Then since A D is perpendicular to B C, the angle A D C is a right angle, and if A E is perpendicular to B C, the angle A E C is a right angle, and the angles A D C and A E C are equal. (*Authority.*) From the point E draw E F parallel to A D; F E C will be a right angle, and equal to A D C, and — (*Hesitates.*)

T. What other angle also must it be equal to? Don't flinch, Julia.

J. It must also be equal to the angle A E C? (*Rather interrogatively.*)

T. Why certainly, according to the supposition. Please go on.

J. But F E C is only a part of A E C; hence the supposition has led to an absurdity and is not correct. Therefore, from a point outside of a line but one perpendicular can be drawn to the line.

T. Very good, Julia, but there is one point that I did not quite feel satisfied with. You drew F E parallel to A D, and then claimed that F E C was a right angle. What right had you for that claim?

J. Why, if F E is parallel to A D, it must be perpendicular to B C.

T. Well, it does seem so, I confess, but mathematicians think it a point requiring proof. For example, see Propositions VII. and VIII. (*All read them over.*) I think, however, that we can get over this difficulty. What is an angle, John?

John. An angle is the difference in direction of two lines originating from a common point.

T. When are two angles equal?

J. When the difference in direction of their sides is the same.

T. What can you say as to this case?

J. (*studying a little*). That as A D and F E are parallel, the difference of direction of D C and D A must be the same as the difference in direction of E C and E F.

T. What then?

J. The angle F E C must be equal to the angle A D C.

T. And to what other angle, John, if the supposition is correct?

J. To the angle A E C. (*John goes on and finishes the demonstration.*)

Albert (*raising his hand*). Can't the absurdity be shown another way?

T. The absurdity of what, Albert?

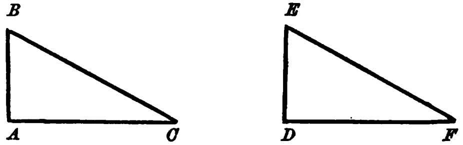
A. The absurdity of the supposition.

T. I think it likely, Albert. Have you a way in mind?

A. Since the angle F E C is equal to the angle A D C, it must be a right angle, and F E must be perpendicular to B C; hence at the point E there are two perpendiculars drawn to B C, which has been proved impossible by the preceding theorem.

T. That is good, and it strengthens the argument that the absurdity of the supposition can be shown from two different points of view.

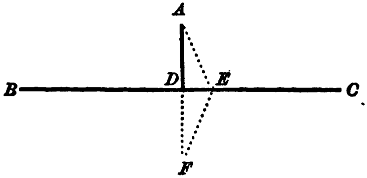
Furthermore, I think we have worked out a valid proof of our theorem, though it is not the usual way of proving it. The usual way is somewhat different, and I want now to develop it; not because it is better than the one we have just taken, but because it illustrates a third method of demonstration, namely, the method by application, which you ought now to understand. In this method figures are proved to be equal or unequal by applying one to the other under specific conditions as to length of lines or size of angles, etc. In using this method, you can suppose a figure folded on some axis, or moved from place to place, if you do not change the conditions given in the theorem. For example, if two triangles have their sides respectively equal and parallel, they are equal.



Let the two triangles A B C and E D F have the A B and D E, B C and E F, and A C and D F respectively equal and parallel, then they will be equal. Now I can suppose one or both of these triangles moved anywhere I please if I keep the sides parallel and do not change their length. Suppose then the triangle A B C applied to the triangle D E F so that A will fall on D, as the side A B is parallel and equal to D E, it will take the same direction as D E, and the point B will fall on E. As A C is parallel and equal to D F, it will take the same direction, and the point C will fall on F. As B falls on E and C on F, the sides B C and E F must coincide. Hence the triangles are equal. Therefore if two triangles have their sides respectively equal and parallel, they are equal.

Now if the conditions had only been that the sides must be equal, I could not only have supposed the triangles moved, but also turned about, for that would not have changed the length of the sides.

Figures also may be folded by supposition, provided always that the specified conditions are not changed. To return to the proposition we have under discussion; we will take the figure as it was constructed after the supposition that a second perpendicular could be drawn.



Thomas, you may go to the board. First prolong the perpendicular A D across the line B C, making D F equal to A D. (*Thomas does so.*) Next connect E F. (*Thomas does so.*) Now, scholars, please note the conditions. What kind of a line is A D F?

All. A straight line.

T. Why, Thomas?

Thomas. Because it is A D prolonged.

T. What are all the angles formed by it with the line B C?

Thos. Right angles.

T. Did we also prolong the supposed perpendicular A E to F?

Thos. Yes, sir. (*All the class agree with him.*)

T. Will you please try to recall the exact directions I gave you, Thomas, for drawing E F.

Thos. (*after some thinking*). You told me to connect the points E and F.

T. Well, is that the same thing as prolonging the line A E? For example, if I should direct you to connect the points B and A, would that be the same as prolonging the line B C?

Thos. No, sir.

T. Then why should you think that connecting the points E and F meant prolonging the line A E?

Thos. We prolonged the line A D.

T. I know we did, Thomas, and I so directed. But I used entirely different language in regard to drawing E F. And yet you seemed to think that the difference was merely accidental, and that I really meant the same.

Now, scholars, I have dwelt upon this point for this reason: it illustrates a fault which possesses all of you, more or less, and some of you exceedingly. It is the fault of not observing and comprehending the exact meaning of statements that are clearly made and well defined. Sometimes, as in the case just discussed, you give it more meaning than was intended, and then in your argument off you go ballooning. Sometimes you fail to see all that the statement conveyed, and then you are at a halt because you can't find the steps by which the argument is advanced. I can account for this only on this supposition, that either you yourselves habitually talk and read so carelessly, or else are in the habit of listening to others who talk so carelessly, that when you come across a statement carefully and exactly worded, you don't appreciate it.

That is, your sense of comprehending good language has become blunted, just as by abuse your sense of sight or hearing might. Now I hope you will take more pains to sharpen that sense, for ability to understand as well as to use language correctly is the supreme test of a cultivated mind.

But to return to our demonstration, if it could be shown that on the supposition that A E is a perpendicular A E and E F would have to be one straight line, what would you say? Ora.

Ora. I would say it was absurd, for there can be but one straight line between two points, and A D F is that line.

T. Well, that would show our supposition false, would n't it?

O. Yes, sir.

T. Can you tell me what evidence would be needed to prove A E and A F a straight line?

O. (*thinking a little*). We would have to prove that the angles A E C and C E F are together equal to two right angles.

T. Do you know anything about either one of those angles?

O. Yes, sir; we know that A E C is a right angle.

T. Provided, Ora.

O. Provided A E is perpendicular.

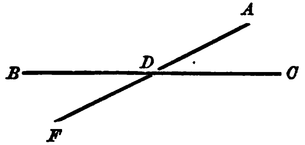
T. Then what do you need to prove?

O. That F E C is or must be a right angle.

T. Can you suggest a way?

O. By folding the figure on the line B C, then F D would fall on A D.

T. Why? (*All look surprised.*) Remember, scholars, that whenever you make an assertion in geometry there should be some reason for it; either an axiom, a definition, or something previously proven. Whenever in an argument you make an assertion and don't accompany it with a reason, you may be sure you are skipping. To show the need of this here, all lines crossing B C would not coincide if folded on the axis B C, as you claim A D and D F will. For example, take a case like this (*drawing*). Here D F would not coincide with A D if folded on B C; and yet you seemed to regard such a coincidence an axiom.



O. The angles A D B and B D F are both right angles, and hence the line F D would take the direction D A.

T. We will stop here to-day; and I want you to complete this demonstration to-morrow, together with the next two propositions.

RECITATION VIII.

T. Anna, you may demonstrate Proposition V. by our first method.

(*Anna goes through the demonstration, showing considerable improvement in her style, but still making omissions that are duly criticised.*)

T. Robert, you may demonstrate it by our second method.

(*Robert begins, but after he gets to the proof, becomes confused and gives up. George is called, who proceeds thus:* —)

George. Fold the figure upon the axis B C. Then, since the angles A D B and B D F are right angles, the line D F will fall on A D, and the point F will come on A; hence—

T. No reason for that last statement, George?

G. Because the lines D F and A D are equal. The line E F will coincide with A E, and the angles A E C and C E F are equal; hence the lines A E and E F form one straight line.

T. There, George, you have jumped. What is your authority for that deduction?

G. If two angles whose sum equals two right angles are placed adjacent to each other, their exterior sides will form one straight line.

T. You did n't state that their sum was equal to two right angles.

G. Since A E C is a right angle, F E C must also be a right angle, and together they will be two right angles.

T. You may go on.

G. Then the lines A E and E F form one straight line. But A F is a straight line, hence there are two straight lines between A and F, which is impossible; hence the supposition is wrong, and there can be but one perpendicular let fall from A on the line B C.

T. Well, George, I have but one criticism to make. You said, "Since A E C is a right angle," when you should have said, "Since by the supposition, A E C is a right angle."

If the teacher thinks best, he can give the class a clew for proving this proposition directly by drawing a second line A E, prolonging A D to F, and connecting E and F as before. Then since A D F is a straight line, A E and E F cannot be, hence the angle A E F is less than two right angles, and half of it, A E D, as shown by application, less than one, etc.

Propositions VI. and VII. will probably be demonstrated with little difficulty, except the usual little omissions and slips, which should be carefully corrected.

The tendency about this stage of a large part of the class to jump details, as has been illustrated, will cause the teacher no little bother, but the defect must be attended to with all possible persistence and patience, and the pupil must be held to a full and accurate argument *quoque temporis dispendio.*

Another fault, too, that will require constant criticising, will be repeating. These difficulties in using the customary text-book are obviated in proportion as the scholar learns the demonstration by rote, but he fails to appreciate the exactness of the language he is memorizing, and loses that discipline in original expression which he so much needs.

How, by a fortunate hit, I succeeded in overcoming to a large extent the former of these faults, I will show in a subsequent lesson.

Another thing that the teacher should now begin to impress upon the scholars more and more is, that when called on, each should go as far as possible with the demonstration without waiting to be questioned and pumped, — a practice at first rather necessary.

Of course a private interview with Robert will follow at some convenient time when the cause of his failure, his method of study, the time devoted to it, and his difficulties will be inquired into, and reproof, encouragement, or assistance will be given, as the teacher deems most needed. You will say that this costs time. Certainly, but what is the teacher for?

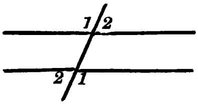
The teacher should also keep the propositions ahead of the class in mind, and sometimes it may be well to make a suggestion in advance as to some construction line, or the method. But at the same time don't make suggestions unless they *are* necessary. When the lesson involves definitions of a more or less arbitrary character, the teacher will find it well to place upon the board, a day in advance, illustrations. For example, suppose the lesson is on the following definitions: —

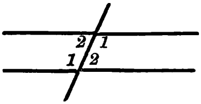
Let two straight lines be intersected by a third.

a. What are the exterior angles?
b. What are the interior angles?
c. What are the alternate exterior angles?
d. What are alternate interior angles?
e. What are external angles on the same side?
f. What are internal angles on the same side?
g. What are opposite external-internal angles?

When this lesson is assigned, the teacher will have placed on the board something as follows: —

42. *c.* Illustrations. *d.* Illustrations.

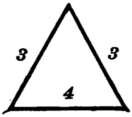


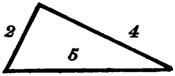


From these illustrations, the pupil should be asked to define the angles.

The same method should be used in defining triangles.

58. *b.* Illustration. *c.* Illustration.



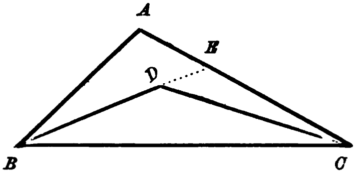


RECITATION IX.

Teacher. Katie, you may take Proposition XIII.

Let me say here in passing, that the numbers of the theorem as above would only be used for the lesson in hand, or as a convenient means of reference. The scholars should not be required to learn the theorems by their numbers. This is not geometry.

Katie. The sum of two lines drawn from any point to the extremities of a line is greater than the sum of any two lines similarly drawn from an included point.



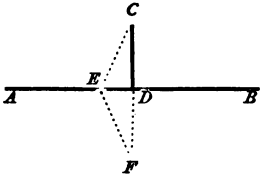
Let A B and B C be two lines drawn from the point A to the extremities of the line B C, and D B and D C be two lines similarly drawn from the included point D, then B A + A C > B D + D C. Prolong B D to E; then D C is shorter than D E + E C. Adding B D to both, B D + D C is less than B D + D E + E C. But B E + E C is less than B A + A C, hence B D + D C is less than B A + A C. Therefore, etc.

T. Well, Katie, there are a few opportunities still left in your demonstrations for criticism. You postulated prolonging A D to E, but you did n't tell us where E was, whether it was in the line A C or very near it; you omitted authorities; you stated by a little jump, which no doubt you thought we would understand, that B E + E C is less than B A + A C; but the oddest thing of all is that you started out to prove that A B + A C is greater than B D + D C, but when you drew your last inference, it was that B D + D C is shorter than B A + A C. Now I know that means the same, but logically, unless you can give some cause for the change, would it not be better to come out as you predicted?

Katie. Yes, sir; I should have said that D E + E C was greater than D C, when I began.

T. Right; and now will you please correct some of the other defects I have called your attention to? (*She does.*) Lewis, you may take Proposition XIV.

Lewis. The shortest distance from any point to a given straight line is a perpendicular to the line. (*Draws figure.*) Let A B be a line, and C a point in the perpendicular C D. Draw any other line, as C E; prolong C D to F, and connect E F. C F is shorter than C E + E F, hence C D or half of C F is shorter than C E, half of C E + E F. Therefore, etc.



T. Well, scholars, I think you have discovered a fourth method of proof.

At this statement, the interest of every pupil was aroused. There is something very fascinating in being discoverers and inventors, and it was easy to see in the faces of all the class the eagerness with which they awaited the revelation and anticipated the honor.

T. (*continuing*). The three usual methods, as you know, are the direct proof, the indirect proof, and proof by application; but you, as I have said, have discovered a fourth, which, as a descriptive name, might be called the hop-skip-and-a-jump method.

The change that came over the pupils' faces, from their late eagerness, to a smile that was half disappointment, as the full meaning of the little joke impressed them, was exceedingly comical. From that time, however, the fault rapidly disappeared, and if any one at any time was inclined to be careless in this particular, a mere allusion to the fourth method of proof was generally sufficient.

When the lesson is assigned that includes such propositions as, for example, the thirty-first of book first, the teacher should have the pupils illustrate them at sight, taking care that the figures are so constructed as to represent each case. In the proposition mentioned, the thirty-first, the third case only will present to the scholars any difficulty, and it will be well to let them study on this for a few days if necessary. Some of my scholars found for this a very much simpler demonstration than any I have ever seen in books.

These illustrations should suffice for the purpose of the writer, which, as has been said, is suggestive, rather than to establish a hard and fast method of presentation. Not only will every teacher have to develop his own method, if he would keep his class alive, but he will have to modify it more or less for every different class and pupil, and in so doing he will find the process delightfully awakening.

CHAPTER III.

RATIO AND PROPORTION. — THE THEORY OF LIMITS.

ALTHOUGH neither ratio nor the theory of limits is properly geometry, yet in the book* to which I have already alluded I have deemed it best to follow the example of authors of geometries in general, and introduce a discussion of them. This I have done in a separate book (Book II.).

This discussion is made positively necessary from the fact that, as arithmetic and algebra are now generally taught, the pupils hitherto get but a rudimentary understanding of ratio, and no knowledge whatever of the theory of limits; yet without a clear comprehension of ratio, little or nothing can be done in geometry; and the theory of limits must be understood, to say the least, in the interest of truth. When the craze for simplifying arithmetic and algebra has had its day, and those branches are taught by teachers who keep in view the training of the faculties as well as the learning of the processes, the study of geometry will be much facilitated.

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* Principles of Plane Geometry.

First, then, have the pupils thoroughly understand what a ratio means, and that $a : b$ and $\frac{a}{b}$ are one and the same thing, and can be used interchangeably at pleasure.

Secondly, that a proportion is merely a way of expressing that certain ratios are equal; and that $a : b :: c : d :: e : f$, etc., means merely that $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, etc.; this, and simply this. If these fundamental points are clearly grasped, everything else about proportions will come easy.

All the changes rung upon proportions are merely the results of the direct application of the axioms to this fundamental fact, and are simply corollaries to it. A great deal of the difficulty that the student in mathematics, especially in arithmetic and algebra, encounters, is because he is allowed to lose sight of the axioms. In demonstrating that in a proportion the product of the extremes is equal to the product of the means, I have often, I might almost say always, been surprised to find that pupils who had been studying algebra for more than a year did not understand the principle by which $ad = bc$ was derived from $\frac{a}{b} = \frac{c}{d}$.

Nor is it enough that ratio, as applied to magnitudes in general, be made clear. Special attention must be given to it in the comparison of the magnitudes of lines. The pupil will quickly see that the ratio between the lengths of two lines five and three feet long respectively is as 5 : 3, or $\frac{5}{3}$, but herein lies the danger; he will cling to the notion that to be commensurable two lines must both be divisible by one of the standard units of measurement, such as the inch, foot, or yard. This notion must be dislodged. The pupil must be brought to see that in comparing lines, every problem has its own unit; and that if the unit is the foot or the inch, it is only because, as in the example above, the lengths of the lines permitted it. He should have numerous problems given him in finding the ratio between lines drawn at random. The usual method is by laying off the shorter on the longer, the remainder on the shorter, etc.; and the process may be profitably compared with a similar way for finding the greatest common divisor in arithmetic. Let me also add here that when he comes, in

Book IV., to the question, What are similar polygons? he should find his answer by having two similar polygons (triangles, for example, but not equilateral) given him, and by applying this process to the homologous sides, he should be led to discover for himself the common ratio. For this work he must use good dividers with jointed legs, and learn to work with exactness. I wonder how many teachers teach their scholars the real purpose of the joints in the legs of dividers.

But to return to our subject, another way to find the common divisor of two lines, and hence their ratio, is as follows: lay off upon the longer A B, the shorter A′ B′; if there is no remainder, A′ B′ is the common divisor. If there is a remainder C B, divide A′ B′ into parts $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., till a part C′ B′ is found equal or less than C B; if equal, it is the common measure; if less, lay it off on C B with the remainder D B. Divide C′ B′ into parts $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., till a part is obtained equal or less than D B. Continue this till a measure is found, or till the lines may be supposed incommensurable. Of course, at this stage, the divisions must be made by measure. I do not suggest this as a better way than the former, for it is not, but as an occasional one, to let the scholars see that if two magnitudes are incommensurable, it does not follow that neither of them is divisible by any possible unit; an impression they are apt to get from the other method. By this process one of the lines, C′ B′, is always divisible by the unit at whatever stage of reduction, and in this respect it agrees with the supposition in all the propositions where incommensurable magnitudes are compared. I shall never forget what perplexity the lack of understanding this little point cost me when taking my first course in geometry.

The pupils will acquire a still clearer and broader comprehension of this subject, if taught to find the common measure and ratio of angles in a similar way to that first given for lines. It will, to be sure, be necessary to anticipate the problem for constructing an angle equal to a given angle (Book III.); but this can easily be done. For every pupil knows that the radii of equal circles are equal, and if the chords connecting their extremities are equal, the triangles thus formed are naturally equilateral, and hence equiangular, as previously demonstrated in Book I.

Lastly, the pupils' attention should be called to a point concerning which probably no question or difficulty would occur to them. And yet it is the point of origin of a controversy extending from the time of Euclid to the present day; a controversy, let me add, the most interesting and instructive of any that the development of the science has given rise to. The ratio of commensurable magnitudes is expressed by the ratio of the numbers indicating the units in each magnitude. But as there are no such numbers of equal units in incommensurable magnitudes, do such magnitudes have an exact ratio? Do they admit of being formed into an exact proportion? Mathematicians seem to have been unanimous in answering these questions in the affirmative, but then came the next question: How can the correctness of such proportions be proven? To the superficial glance, this question, too, seems harmless enough and, perhaps, simple enough, but it has first and last caused "much throwing about of brains." The theories for demonstrating this have for two thousand years constituted, perhaps, the sole regard that forbade mathematics the right to be called an exact science. The solution of this difficulty is the great mathematical achievement of the present century. M. Duhamel,* in his *Cours de mécanique de l'école polytechnique*, published in 1845, was the first clearly and fully to expound the principle which not only solved in a rational way the relation of incommensurable magnitudes, but dispelled the mystery that hitherto surrounded the operations of the calculus, making them seem more like a work of legerdemain than of reason.

In accrediting the discovery of the theory of limits to M. Duhamel and the present century, I would be understood in the sense only in which the honor is usually awarded for every great discovery, — to the man and the time that gave it the finishing touch.

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* M. Jean Marie Constant Duhamel, the great French mathematician, was born in St. Malo in 1797, and died in Paris in 1872. He was a graduate of L'École Polytechnique, and teacher in the same institution for nearly fifty years. In 1840 he became a member of L'Académie des Sciences, and in 1851 was chosen titular professor of the faculty of sciences. Besides his Calculus, he published *Cours de mécanique de l'école polytechnique* (2 vols., 1845), and *Des méthodes dans les sciences de raisonnement* (1866).

The theory of variables and their limits was well understood by the ancient mathematicians, and formed the basis of the *method of exhaustion*, used by Euclid and Archimedes, two hundred and fifty years B. C., and were formulated by them to a point all ready for the application made of them by Duhamel.*

But as Euclid and Archimedes left them, so practically they remained for nearly twenty centuries, the reverence paid by succeeding mathematicians to the authority of those great men protecting their work against all progress.

The intellectual and mathematical revival of the seventeenth century found two men, Newton and Leibnitz, who dared to amend the venerable *method of exhaustion* by giving it a more positive explanation, but in both cases they changed it into an absurdity, as will be shown hereafter. But as in the case of Euclid and Archimedes, so reverence for the authority of Newton and Leibnitz hallowed even their errors, and for more than two hundred years, mathematicians, blinded by authoritative but absurd assumptions regarding infinity and infinitesimals, refused to see the plain and simple solution. At first Duhamel's exposition made but little progress, especially in England and America, and its elevation of late to popular favor may rightly be regarded as a new departure. I will venture to assert that few teachers ever heard of it in either the training school or in college. The *theory of limits* involves ideas and conceptions that are new to the pupils also, and will for the most part have to be explained to them. Hence it is of prime importance that the teacher himself clearly understands it. For this reason I have deemed it best to address the instruction on this topic directly to the teacher who may need it; leaving it for him to digest it and present it to his pupils as he deems best.

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* La notion des infiniment petits remonte à Archimède; elle s'est présentée d'elle-même dans la mesure des grandeurs géométriques, aussitôt qu'on a voulu considérer les aires de courbes différentes du cercle, ou les volumes de corps terminés par des surfaces courbes autres que celles du cylindre, du cône et de la sphère. Déjà pour la comparaison des volumes de ces derniers corps, ainsi que pour la mesure du cercle, on avait été amené à la conception fondamentale des limites. Ainsi les deux idées généralles les plus fécondes des sciences mathématiques remontent presque jusqu'à leur berceau et doivent, par conséquent, se présenter presque au début de leur enseignement.—*Éléments de calcul infinitésimal,* Vol. I., p. xv.

Up to this point in our mathematical studies, we have been dealing with quantities as fixed, that is, as remaining the same throughout each problem or discussion. For example, what will six pounds of sugar cost at eight cents a pound? Though in another problem the quantity and price may both be different, in this one they are fixed at six pounds and eight cents respectively, and the answer may be exactly found. Such quantities are called constants. From our early mathematical training we have come to look upon all quantities as constants, but in truth, our daily experience brings to us problems where the quantities are not so. If a grocer were pouring sugar worth eight cents a pound from one barrel into another, and you were asked to find the value of the sugar in each barrel while the process was going on, you would find yourself called to face a new problem; a problem where one of the magnitudes was continually changing, increasing in one barrel and decreasing in the other. Again a traveller on a rapidly moving express train glances at a milestone, and says, "We have gone just twenty miles." Even if it were exactly true as he began to speak, it had ceased to be before he ended. He was dealing with varying magnitudes, the distance the train had gone increasing, and the distance to the terminus decreasing. Such magnitudes are called variables; and the very nature and fundamental conception of them is, that constantly changing, they cannot be exactly measured or expressed. It will go hard, however, if mathematics don't make some use of them. The peculiarity of the variables above described is, that after a while the changing ceases: the sugar is all in one barrel, and the express train has reached its destination; the increasing variables have reached each its maximum, and the decreasing each its minimum, the latter in the instances above being zero.

In a circle fix a point anywhere not at the centre, and through it draw a chord. Suppose this chord revolved on the fixed point, but always continuing a chord. The chord itself as a whole, or either of its segments, will be good illustrations of alternately increasing and decreasing variables, each at a certain point attaining its maximum, and at another its minimum. In many text-books of geometry such variables are specially treated under maxima and minima, and are a prominent feature of some parts of higher mathematics.

But there is another class of variables that differ from these just described, in this, that they never reach a maximum or a minimum; they must be conceived as forever increasing or decreasing under laws that do not permit the variations to stop. For example, suppose an inscribed polygon to have the number of its sides continually increasing from, say, a triangle to an inscribed quadrilateral, pentagon, hexagon, etc., indefinitely. First we observe that the perimeter of the inscribed polygon is an increasing variable, approaching the circumference of the circle. Will it ever reach it? By the long prevailing theory of infinitesimals, we are taught that in time by the continual dividing, the sides become so infinitely short that they coincide with their arcs, and therefore the perimeter coincides with the circumference. It is a lie with extenuating circumstances; that is, its promulgators do not intend to mislead, but they themselves have been misled and don't know any better. As we are all so liable to be guilty in this way, we can't afford to be uncharitable. The perimeter of the polygon will never become a circumference, for no dividing of a straight line will ever make its parts arcs. It will approach the circumference forever, approach it within an infinitely small difference, but the difference will still be a magnitude and not zero. It is impossible to express the exact value of the perimeter thus forever changing. All we know with certainty about it is that it is forever approaching but will never equal the circumference of the circle. The magnitude thus approached by a variable is called the *limit* of the variable. The difference between a limit and a maximum or a minimum is that the maximum or minimum may be attained; that is, they are values of the variable; but the limit never. The variable, too, when it reaches its maximum or minimum, may become a constant, as the distances measured from the express train, or the amount of sugar in the barrels; or it may become a variable of an opposite tendency, as the revolving chord. But the variable that is approaching its limit never reaches it, however infinitesimal the difference may become, and yet never stops approaching, never becomes a constant, never reverses. It is perpetual motion in magnitude.

The area, too, of the inscribed polygon in the case cited above is an increasing variable approaching forever as its limit the area of the circle. The difference between the area of the polygon and the area of the circle is a decreasing variable approaching indefinitely its limit zero.

A variable approaching a limit may be of such a character that if we supposed the process reversed we would have a variable approaching a maximum or a minimum. For example, if in the illustration above of the inscribed polygon, we suppose the polygon to be an equilateral triangle, changed by doubling the sides to a regular inscribed hexagon, duodecagon, etc., all the facts would be the same as already explained; but if at any point we supposed the process reversed, the area and perimeter would become decreasing variables eventually reaching their minima, the area and perimeter of the inscribed equilateral triangle. But these are altogether different variables.

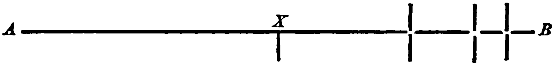
Let us, in the light of what has now been said, investigate the so-called sum of an infinite geometrical series. The formula given us for it is $S = \frac{a}{1-r}$. Applying it to the infinite series 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, etc., we are told the sum is 4; another of those stories with extenuating circumstances. By its very nature, an infinite series is forever growing, forever extending. The moment it would stop it would become a finite, not an infinite series. Try to express the sum of such a series and you would find it even more elusive than the exact distance of a moving train.

> "Comprensa manus effugit imago
> Par levibus ventis, volucrique simillima somno."

To call any fixed magnitude the sum of such an expanding series would be absurd in any science, to say nothing of an exact science. The sum of an infinite series is a variable, and $\frac{a}{1-r}$ is not its sum at all, but the limit towards which the sum is forever approaching but never to reach. We may use it for the sum, because it is the nearest constant quantity to it that we can get, on the same principle as we use 3.1416 for the value of π, because it is a convenient approximate.

In a right triangle suppose one leg forever shortening. This leg would be a decreasing variable, approaching its inferior limit zero. It can never reach zero, for if it did, the figure would cease to be a right triangle, which would not accord with the hypothesis "in a right triangle." While the leg, therefore, is decreasing, it must be by some law that does not annihilate the triangle. The hypotenuse too is a decreasing variable approaching, as its inferior limit, the length of the constant leg, which it cannot reach for the reason given above. The area is a decreasing variable approaching its inferior limit zero. The angle included by the hypotenuse and the constant leg is a decreasing variable approaching its inferior limit zero. The angle included by the hypotenuse and the variable leg is an increasing variable approaching a right angle as its superior limit.

One more illustration. On the line A B, suppose a point X moving in such a way that it goes the first unit of time one half the



distance A B, in the second unit of time one half the remainder, in the third unit of time one half the remainder, and so on forever, when will it reach B? Never. The distance A X is an increasing variable indefinitely approaching the magnitude A B, but it can never equal it, and the distance X B is a decreasing variable indefinitely approaching zero, but it will never become zero, though it may become infinitesimally small.

The ancient mathematicians discovered and taught the nature of variables and their limits to be as I have explained them; but they timidly ventured the supposition that somewhere in infinity the variable *exhausted* the difference between itself and its limit, hence the name, *method, or theory of exhaustion.* Neither Euclid nor Archimedes, and of course none of their followers, seem to have made any use of this supposition in demonstrating theorems involving variable magnitudes.* Narrowed in their field of research, but most rigorously logical, they proved these theorems by a more roundabout but unanswerable demonstration, as will be shown hereafter.

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* Let me acknowledge here my great indebtedness in many of these particulars to "The Philosophy of Mathematics," by Prof. A. T. Bledsoe; published by J. B. Lippincot Company, Philadelphia, 1886. I would recommend this book to all students who may wish to further investigate the subject in hand.

Newton, too, accepted the idea of variables and their limits, as expounded by the ancients and explained in these pages, but in his theory known as *fluxions* he boldly assumed that in the magic of infinity, in some way the variable flowed into its limit and became equal to it.

"Tantum ævi longinqua valet mutare vetustas."

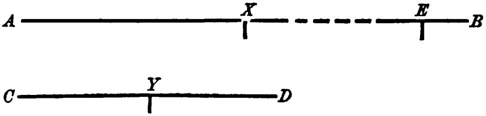
Thus stated, Newton's *fluxions* do not really differ from the *method of exhaustion* of the ancients, except that he substituted boldness for their timidity, and confidently based conclusions on his assumption. Less logical but more positive.

Leibnitz rejected both the idea of "exhaustion" and "flowing into," and adhered more firmly to the fundamental idea of variables and their limits, but held that when the difference between them becomes infinitely small, it may be considered zero and disregarded; so that his conclusion was practically the same as the others. These conclusions may be summed up thus: Variables never reach their limits though they approach the latter indefinitely, but in infinity, by exhausting the difference, or by flowing into their limits, or by coming so near to them that the infinitely small difference may be regarded as zero, the variables become equal to their limits. There was practically no difference in these three theories, and as I have stated them, their absurdity would seem to be sufficiently clear. All the demonstrations or explanations that have been offered for either of them may be summed up as amounting to this: "You can't tell what may happen in infinity." Of the three the theory of Leibnitz became the most popular, for to throw them away seemed the easiest way to dispose of such little things.

The theories of Newton and Leibnitz might have held their ground longer than they have, if their followers had rested on the authority of their great masters, and kept still. But on the contrary, they felt called upon to explain and defend, and the more they explained and defended, the more they disagreed among themselves as to the true explanation and defence, and the more apparent to thinkers became the absurdity.

But to return to the *theory of limits*, it remains for me to complete the exposition of it, so far as it is needed, in the demonstrations of plane or solid geometry, to point out the fundamental principles deduced from it, and to illustrate its working in the demonstration of propositions otherwise requiring more roundabout or intricate methods. By comparing it with other methods, I hope to show its advantage over them. The fundamental proposition established from variables is this: "*If two variables, as they approach their limits indefinitely, have a constant ratio, their limits will have the same ratio.*"

Let A X and C Y, formed by the moving points X and Y, be two variables having to each other the constant ratio *m* as they indefinitely approach their respective limits, A B and C D; then the limits will have to each other the same ratio.



For if A B is not *m* times greater than C D, it must be either greater or less than *m* times. Suppose it greater, and that *m* times C D equals A E. Now while the var. C Y can never have a value equal to its lim. C D, although indefinitely approaching it, the var. A X may have values between A E and A B, and therefore it will be more than *m* times greater than the var. C Y; but this is contrary to hypothesis; hence A B cannot be more than *m* times greater than C D. In the same way it may be shown that A B cannot be less than *m* times C D; therefore the limits have the same ratio as the variables indefinitely approaching them.

I have appended to this proposition as a corollary,* what is generally given as the fundamental principle itself, and so stated by M. Duhamel.†

If two variables, as they indefinitely approach their limits, are constantly equal, their limits are equal.

It is evident on a moment's thought that this is merely supposing the variables to have the constant ratio 1, and is therefore only a special case under the more general principle as I have given it. It is not even necessary as a corollary, but I have thought best to so insert it. For the theory of limits is, as I have said, a new experience to the pupils, and their digestion of it may be aided by an otherwise superfluous expansion.

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* Principles of Plane Geometry, page 26, § 138.

† Principe Fondamental des Limits. Si deux quantités variables sont constamment égales et tendent chacune vers une limite, ces deux limites sont nécessairement égales. — *Éléments de calcul infinitésimal,* Vol. I., p. 13.

The corollary can be proven, as the main proposition has, by supposing the variables indefinitely approaching the lines as their limits, to be equal; but variety also aids digestion, and I would suggest something like the following:—

Let the equal variables be two regular inscribed polygons, which, by the continual and indefinite doubling of the number of their sides, are continually and indefinitely approaching the areas of their respective circles as their limits. If the circles are not equal, one of them must be the larger. Then the sides of the inscribed polygon, at every stage of approximation, must be greater than the homologous sides of the other, and the polygons will be unequal; which is contrary to hypothesis, which requires them to be equal; hence the circles must be equal; therefore, if two variables, etc.

The importance of this fundamental principle of the *theory of limits* thus established by *reductio ad absurdum*, that if two variables indefinitely approaching their limits have a constant ratio, the limits have the same ratio, cannot be too strongly impressed, and the principle itself cannot be too clearly grasped. I would advise the teacher to turn it over and over, and to apply it, and to study it on all sides even after he thinks he understands it.

Two variables may not have a constant ratio, that is, their ratio itself may be a variable, and the problem may be to find its limit. For example, the ratio of the hypotenuse to the variable leg, in the case supposed above, is a variable increasing without limit as the variable leg approaches its limit zero, and the ratio of the hypotenuse to the constant leg is a variable ratio approaching as its limit unity.

There is one other combination of variables that I have thought best to dignify as a separate proposition, although it is hardly more than a corollary of the main principle.

If of variables approaching their limits indefinitely, the product of any two equals a third, the product of their limits will equal the limit of the third, or in other words, the product of the limits of two variables is the limit of their product.

It is clear that as the product of two variables constantly equals a third, it is a variable itself, and we have two variables constantly equal as they approach their limits.

In the same way it may be shown that the limit of the quotient of two variables is the quotient of their limits; that the sum of the limits of any number of variables is the limit of their sums; that the difference of the limits of variables is the limit of their difference; and similarly for powers and roots: always provided that the variables are such that they approach their limits indefinitely.

To illustrate the working of the *theory of limits*, let us first take the following proposition: —*

In the same circle, or in equal circles, angles at the centre are to each other as their arcs.

CASE I. *When they are commensurable.*

CASE II. *When they are incommensurable.*

To prove Case I., we divide both arcs into small arcs, each equal to the common measure, and draw radii to the points of division, thus dividing the angles at the centre into a corresponding number of equal angles. From this we establish the equality of ratio. But our chief interest is with Case II.

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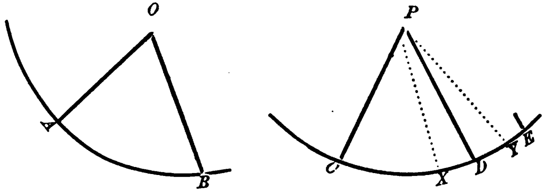
* Principles of Plane Geometry, Book III., Proposition XVII.

Let the arcs A B and C D be incommensurable, still they will be in proportion to the angles at the centre. Suppose a unit of measure to be obtained by subdividing the arc A B, and that this unit laid off on the arc C D gives the arc C X, and a remainder X D. Draw the radius X P. Since the arcs A B and C X are commensurable, the $\angle AOB : \angle CPX :: \frown AB : \frown CX$. By making the unit smaller and smaller by successively subdividing, we may bring the point X nearer and nearer to the point D indefinitely, the proportion just given holding for every stage of approximation.



Before finding the demonstration by the theory of limits, let us see how it would have been treated by the other methods. Newton's followers would claim that the variable arc C X eventually in infinity flowed into and became equal to the arc C D; Leibnitz's followers would say that the difference eventually became so small that it might be considered zero and disregarded; logically, according to both, incommensurable magnitudes are commensurable, and there is really no Case II.

The ancients and their followers would have *supposed* that in time the difference X D of the arcs became exhausted, but they would have rested no argument on that supposition. They would have proceeded to demonstrate the proposition by *reductio ad absurdum*, the method, let it be observed, by which we have demonstrated the fundamental principles of the theory of limits. The argument would have been somewhat as follows:—

If the angle A O B is not to the angle C P D as their arcs, then they will be to each other as the arc A B is to an arc greater or less than C D; let us suppose greater, and that C E is the arc. Suppose the arc A B divided into parts smaller than D E. Laying off equal parts on the arc C D, beginning at C, one point of division at least will come between D and E; let Y be that point, and draw the radius P Y. Since the arcs A B and C Y are commensurable, by Case I.,

$$\angle AOB : \angle CPY :: \frown AB : \frown CY.$$

But by supposition

$$\angle AOB : \angle CPD :: \frown AB : \frown CE.$$

Hence since the antecedents are the same in both proportions,

$$\angle CPY : \angle CPD :: \frown CY : \frown CE.$$

That is, the angle C P Y is to an angle smaller than itself, as the arc C Y is to an arc greater than itself, which is absurd, etc.

By the *theory of limits*, the argument will be as follows: Commencing at the proportion

$$\angle AOB : \angle CPX :: \frown AB : \frown CX,$$

we continue; the arc C X is a variable indefinitely approaching its limit C D, and the angle C P X is a variable indefinitely approaching its limit C P D; as these variables are always in proportion to the constants, the arc A B and the angle A O B, their limits will be in proportion to the same constants.

If two variables are constantly equal, etc., therefore

$$\angle AOB : \angle CPD :: \frown AB : \frown CD.$$

Here are no contradictory suppositions or assumptions; no positive quantities to be called zero; no painful, roundabout *reductio ad absurdum*, but a clear, brief, direct, and impregnable proof.

The advantage of the method of proof by the *theory of limits*, over the ancient method, is this: that the fundamental principles of the *theory of limits* is first established by *reductio ad absurdum*, and afterwards used briefly and directly to prove the propositions involving variables. While in all such cases, as in the proposition above, and again, where it is demonstrated that the area of a rectangle is equal to the product of the base by the altitude, and elsewhere, the ancients had to resort every time to the *reductio ad absurdum*. It was something as if we should omit from Book I. the proposition: If two sides and included angle of one triangle are equal to two sides and included angle of another, the triangles are equal; and every time the principle occurred, re-demonstrate it.

The advantage of the method by the *theory of limits*, over those of Newton and Leibnitz, is that it is true.

Let us next take the following proposition: —

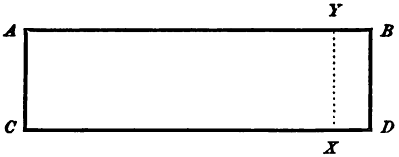
The area of a rectangle is equal to the product of its base and altitude.

This is first established in the case where the base and altitude are commensurable. It need only be added, that the number of linear *units* in the base cannot be multiplied by the number of linear *units* in the altitude, and even if they could be, the product would not be surface units. What we do is to multiply together two abstract numbers, each equal, it is true, to the number of linear units in one of the aforesaid lines, *but not those units;* and the product is an *abstract* number, which we must prove will always be equal to the number of surface units in the rectangle, constructed on the same scale as the linear units. Make this understood.

CASE II. — *When the base and altitude are incommensurable.*

Still the area is equal to C D × A C.

Suppose a unit of measure to be obtained by dividing A C into a number of equal parts, and commencing at C this unit be laid off the greatest number of times on C D, giving C X, and the remainder X D. Erect the perpendicular X Y. Since A C



and C X are commensurable, the area of $ACYX = AC \times CX$. By diminishing the unit, the point X can be made to approach indefinitely the point D. Then $AC \times CX$ is a variable approaching indefinitely the limit, $AC \times CD$, and the area of the rectangle A C X Y is a variable approaching indefinitely the area of the rectangle A C B D; and as these variables are constantly equal, their limits are equal; therefore, etc.

Lastly: *the area of a circle is equal to the product of the circumference by half the radius.*

Suppose a regular inscribed polygon; the area of this polygon as previously demonstrated is equal to half the product of its perimeter and apothem.

Suppose this polygon, by continually doubling the number of its sides, becomes successively a regular inscribed polygon whose sides are multiplying indefinitely. Its perimeter is a variable approaching indefinitely the circumference, and its apothem is a variable indefinitely approaching the radius, and half of their product is constantly equal to the area of the polygon, which is a variable indefinitely approaching the area of the circle; hence half the product of the limits of the two first must equal the limit of the second; hence, the area of the circle is equal to its circumference multiplied by half the radius.

CHAPTER IV.

OPINIONS OF EDUCATORS.

I HAVE said that the method of teaching geometry which has been advocated in these pages is in accord with the opinions of all eminent educators. In this, the concluding chapter, I have thought it well to cite a few of those opinions. Of teachers who may read this book, I have only this to ask: first, that they determine for themselves whether or not these opinions are sound; secondly, that if they consider them sound, they trace them to their logical conclusion in theory and practice; lastly, if that conclusion agrees with mine, or whatever it may be, that they make a consistent and determined effort to carry it out.

KING SOLOMON.

Wisdom is the principal thing; therefore get wisdom; and with all thy getting, get understanding. — Prov. iv. 7.

EMMANUEL KANT.

The best way to comprehend is to do. What we learn the most thoroughly is what we learn to some extent by ourselves.

CONDILLAC.

I grant that the education which cultivates only the memory may make prodigies, and that it has done so, but these prodigies last only during infancy. . . . He who has not learned to reason, has not been instructed, or what is worse, has been badly instructed.

OSCAR BROWNING.

The second great vindicator of naturalistic education, Montaigne, is more outspoken and more consistent . . . Although his precepts are not systematic, and are thrown out rather as hints for reflection, yet there is no doubt that they exercised a very important influence both upon Locke and Rousseau. Like Rabelais he was profoundly dissatisfied with the pedantry of his time. "To what use serves learning, if the understanding be away?" He says of the scholars of the age, "Whosoever shall narrowly pry into and thoroughly sift this sort of people, wherewith the world is so pestered, will, as I have done, find that, for the most part, they neither understand others nor themselves, and that their memories are full enough it is true, but the judgment totally devoid and empty." The dialectics of that age stuffed the heads of its pupils full of barren knowledge, ill digested, which weighed down the mind without developing it. Philosophy had hardened into a number of dry formulæ, which were to be learned by heart, and as Montaigne says, *Savoir par cœur n'est pas savoir*"; To learn by rote is no true knowledge. He is particularly alive to the danger of useless erudition. "Too much learning stifles the soul, just as plants are stifled by too much moisture, and lamps by too much oil."—*Educational Theories.*

T. TATE.

The mind, from its very constitution, seeks to develop itself. A boy is not a mere recipient of knowledge; his faculties are continually developing themselves by exercise. Everything in the world around him tends to stimulate this development. His Creator has placed him in this beautiful world, where all its laws and phenomena tend to quicken, develop, and elevate his physical, intellectual, and moral faculties. The creature should surely follow out the intentions of the Creator!

But educators, in the place of fostering this development, have too frequently directed their energies to counteract it; instead of regarding knowledge as a *means*, they have looked upon it as an *end*. "Some propose," observes Woodbridge, "as the object of all their efforts, to communicate as much positive knowledge as possible; they often produce living encyclopædias, unfit for useful activity. Others perceive how little this accumulation of abstract knowledge avails in preparation for active life, and direct their attention almost exclusively to matters of a practical nature. On this plan there is no small danger of producing mere instruments for others, — men almost incapable of original thought or independent action."

These systems, taken separately, are obviously imperfect. . . . Without losing sight of the importance of practical knowledge, especially at the later stages of elementary instruction, the truly enlightened educator will ever regard the development of the faculties as the great end of all his teaching; from the various useful matters of instruction he will always select that which is best calculated to secure this end, and his mode or system of teaching will always have a reference to the same great end. The question with him will not be, Have I conveyed the greatest amount of technical knowledge in the least possible time, have I engrafted the ideas of the man upon the mind of the boy? But it will rather be, Have I awakened any element of intellectual or moral vitality which had hitherto lain dormant? Have I invigorated or purified any faculty which had hitherto existed in a feeble or in an imperfect state of development ? — *Philosophy of Education.*

HERBERT SPENCER.

When the observing and inventive faculties have attained the requisite power, the pupil may be introduced to empirical geometry ; that is, geometry dealing with methodical solutions, but not with the demonstrations of them. . . . Having thus helped him to the solution of his first problem, with the view of illustrating the nature of geometrical methods, he is in future to be left altogether to his own ingenuity in solving the questions put to him. To bisect a line, to erect a perpendicular, to describe a square, to bisect an angle, to draw a line parallel to a given line, to describe a hexagon, are problems which a little patience will enable him to find out. And from these he may be led on to questions of a more complex kind ; all of which, under judicious management, he will puzzle through unhelped. Doubtless many of those brought up under the old regime will look upon this assertion sceptically. We speak from facts, however, and those neither few nor special. We have seen a class of boys become so interested in making out solutions to these problems, as to look forward to their geometry lesson as the event of the week.

Within the last month we have been told of one girls' school in which some of the young ladies voluntarily occupy themselves with geometrical questions out of school hours; and of another, in which they not only do this, but in which one of them is begging for problems to find out during the holidays—both which facts we state on the authority of the teacher. There could, indeed, be no stronger proofs than are thus afforded, of the practicability and immense advantage of self-development. A branch of knowledge which, as commonly taught, is dry and even repulsive, may, by following the method of nature, be made extremely interesting and profoundly beneficial. We say profoundly beneficial, because the effects are not confined to the gaining of geometrical facts, but often revolutionize the whole state of mind. It has repeatedly occurred that those who have been stupefied by the ordinary school-drill—by its abstract formulas, by its wearisome tasks, by its cramming—have suddenly had their intellects roused, by thus ceasing to make them passive recipients, and inducing them to become active discoverers. The discouragement brought about by bad teaching having been diminished by a little sympathy, and sufficient perseverance induced to achieve a first success, there arises a revulsion of feeling affecting the whole nature. They no longer find themselves incompetent; they too can do something. And gradually as success follows success, the incubus of despair disappears, and they attack the difficulties of their other studies with a courage that insures conquest. . . . But the making education a process of self-evolution has other advantages than this of keeping our lessons in the right order. In the first place, it guarantees a vividness and a permanency of impression which the usual method can never produce. Any piece of knowledge which the pupil has himself acquired comes, by virtue of the conquest, much more thoroughly his than it could else be. The preliminary activity of mind which his success implies, the concentration of thought necessary to it, and the excitement consequent on his triumph, conspire to register all the facts in his memory in a way that no mere information heard from a teacher, or read in a school book, can be registered. Even if he fails, the tension to which his faculties have been wound up insures his remembrance of the solution when given him, better than half a dozen repetitions would. Observe again that this discipline necessitates a continuous organization of the knowledge he acquires. It is the very nature of facts and inferences, assimilated in this normal manner, that they successively become the premises of further conclusions, — the means of solving still further questions. The solution of yesterday's problem helps the pupil in mastering to-day's. Thus the knowledge is turned into faculty as soon as it is taken in, and forthwith aids in the general function of thinking, — does not lie merely written in the pages of an internal library as when rote-learnt. — *Education.*

The following comments on a discussion that followed a paper on teaching geometry, read at the last meeting of the Massachusetts High and Classical School Teachers' Association, are taken, by permission of the writer, from the *Journal of Education* of April 18, 1889. The writer's name, at his own request, is withheld, but he will permit me to say that he is an earnest advocate of reform in teaching mathematics, and the author of an arithmetic designed to promote it: —

Mr. J. W. MacDonald's paper on this subject, and the interesting discussion that followed it, at the recent meeting of the Massachusetts Association of Classical and High School Teachers, indicate that there are a few teachers, at least, who are willing to follow common-sense and sound laws of pedagogy, rather than school-room and text-book tradition. The discussion also indicates that a large majority of teachers are not yet willing to use in the school-room the same sort of intelligence and shrewdness that they use in the management of their own families, or that they continually see displayed in all kinds of business.

The question at issue between Mr. MacDonald and his opponents — so far, at least, as the regular definitions and propositions are concerned — appears to be whether it is better for a pupil to gain an experience of his own, or to learn from a textbook the experience of some one else; whether, for example, it is better for him to make the definition of a plane the result of his own observation and thought, or to accept it as the result of study on the part of his teacher or text-book maker.

Many teachers think that progress by Mr. MacDonald's method is too slow. Let us consider this question for a moment. The two objects of the study of geometry are: (1) a knowledge of the successive steps that must be taken in order to find the volume of a sphere; (2) mental power enough to enable the pupil to reason accurately and conclusively in regard to each step, and to solve at each stage of his progress such geometrical problems as can legitimately be considered to depend for their solution on principles already learned.

Mr. MacDonald's opponents try to attain the first object by making their pupils study the definitions and clearly demonstrated propositions of the text-book, and the second by supplying a large amount of original work, the greater part of which is generally done by only the brightest pupils. Mr. MacDonald tries to attain both objects at once by making all the work original, and by directing it mainly toward the necessary steps in the chain of reasoning that he wishes to establish.

It has been said that Mr. MacDonald's method will work only with the brightest pupils; my own experience tells me otherwise. I once tried it successfully with one of the most hopeless pupils I ever had, and I believe it is the only method by which I could have given him enough knowledge and power to enable him to pass his examination.

Sometimes I think that we *professional* teachers are the worst teachers; a non-professional teacher certainly appears to produce more lasting results. If a boy goes into a counting-room, he very soon learns from those over him principles of business that he never forgets; if he goes into the woods with an Indian guide, the Indian — unable perhaps to read or write — will immediately put in practice the very laws of pedagogy so ably advocated by Joseph Payne, Compayré, Herbert Spencer, Rousseau, Pestalozzi, and others, and the result is a real acquisition of knowledge and power gained in an incredibly short time. If this same boy goes to school or college and studies geometry, or logic, or history, the chances are that six months after he has dropped a subject it has entirely gone from him. I do not believe that the reason for this lies so much in a lack of taste for the subject, as in the absence of a proper method of instruction. Let the pupils do most of the work here, as in the other cases referred to, and the result will amount to something. I once studied logic for six months, and all that I remember of it now is the word *barbara*. I once went skating with a friend who taught me to make the "grape-vine," and years afterward, although I had not skated in the mean while, I found that I could make the "grape-vine." I once took a year's course in analytic geometry, and did fairly well in it; a year after dropping the subject, I was called upon to teach it, and found to my dismay that my previous work in it had left the same impression on my mind that is made by drops of water on a duck's back. I have no doubt that a single month's work by Mr. MacDonald's method would have produced a lasting result — small, perhaps, but nevertheless real — that would have been a benefit to me as a teacher.

In our teaching, are we not apt to be satisfied if we get our pupils over the prescribed ground understandingly in the time allowed? And in doing this do we not, in the majority of cases, fail to cause them to make a single real acquisition of lasting benefit? It may be said that no pupil can study a subject understandingly without increasing his mental power, even if he does forget what he has learned. This is doubtless true, but is n't life too short for such work, when the same amount of mental power plus a permanent knowledge can be obtained in less time?

I beg leave to quote here a few pedagogical principles which the good common-sense of all teachers compels them to assent to, but which school-room tradition, or a mistaken idea of progress, prevents most teachers from practising: —

"Our pedantic mania for instructing constantly leads us to teach children what they can learn far better for themselves, and to lose sight of what we alone can teach them.

"Let him know a thing because he has found it out for himself, and not because you have told him of it. Let him not learn science, but discover it for himself. If once you substitute authority for reason, he will not reason any more; he will only be the sport of other people's opinions." — *J. J. Rousseau.*

"If the grounds of an opinion are not conclusive to the person's own reason, his reason cannot be strengthened, but is likely to be weakened by his adopting it." — *J. S. Mill.*

"It is a truth very imperfectly recognized by teachers, that the education of a youth depends not only on what he learns, but how he learns it; and that some power of the mind is being daily improved or injured by the methods which are adopted in teaching him." — *J. G. Fitch.*

"To give the net product of inquiry, without the inquiry that leads to it, is found to be both enervating and inefficient. General truths, to be of due and permanent use, must be earned." — *Herbert Spencer.*

"The teacher has recognized his true function as simply a direction of the mental machinery which is, in fact, to do all the work itself; for it is not he, but his pupils, that have to learn, and to learn by the exercise of their own minds.

"Personal experience is the condition of development, whether of the body, mind, or moral sense. What the child does himself, and loves to do, forms his habit of doing, but the natural educator, by developing his powers and promoting their exercise, also guides him to the formation of right habits.

"Education can only be gained by *doing a little well*." — *Joseph Payne.*

". . . I am convinced that the method of teaching which approaches most nearly to the method of investigation, is incomparably the best." — *Edmund Burke.*

I sincerely hope that Mr. MacDonald will continue the good work he has begun; and if he does believe that slowly but surely numerous converts will flock to his standard.

Such are the views of men of all times, prominent as educators and thinkers, and it seems to me that it behooves teachers either to refute them or carry them out in practice.